Lecture IV: Basic Isomorphism Theorems

Last Time: • Defined (normal) subgroups generated by a set.

- Left cosets $G/H = \{xH : x \in G\}/\sim$ $\qquad x \sim y \Leftrightarrow x^{-1}y \in H$
- Right — $H\backslash G = \{Hx : x \in G\}/\sim'$ $\qquad x \sim' y \Leftrightarrow xy^{-1} \in H$
- **Thm**: If $H \triangleleft G$, then G/H is a group under $gH * g'H = gg'H$ & $G \twoheadrightarrow G/H$ is gp hom with $\operatorname{Ker} G = H$.
- Cyclic groups & their classification $\begin{cases} G \text{ infinite} \simeq \mathbb{Z} \\ G \text{ finite} \simeq \mathbb{Z}/n\mathbb{Z} \\ (n = |G|) \end{cases}$
- first counting lemma.

TODAY: Discuss 3 Isomorphisms in group Theory.

§1 First Isomorphism Theorem:

Theorem 1: Let G, G' be two groups and $\varphi : G \to G'$ be a group homomorphism. Write $K = \operatorname{Ker}(\varphi) \triangleleft G$ & $H' = \operatorname{Im}(\varphi) < G'$.

Then we have a commutative diagram:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & G' \\ G/K \arrow[r, "\overline{\varphi}"'] \arrow[ur, "\exists \Psi"] & H' \arrow[u, "i"'] \end{tikzcd}$$

$(\Psi(gK) = \overline{\varphi}(gK) = \varphi(g))$

Here: π = natural projection & i is a natural inclusion.

Moreover, $\overline{\varphi}$ is an isomorphism

Proof: Define $\Psi : G/K \to G'$ by $\Psi(gK) = \varphi(g)$

Claim 1: Ψ is well-defined $(g_1K = g_2K \overset{?}{\Longrightarrow} \varphi(g_1) = \varphi(g_2))$

Pf/ $g_1K = g_2K \Leftrightarrow g_2^{-1}g_1 \in K$, so $\varphi(g_2^{-1}g_1) = \varphi(g_2)^{-1}\varphi(g_1) = e'$

Thus: $\varphi(g_1) = \varphi(g_2)$ ✓

Claim 2: Ψ is a group homomorphism:

Pf/ $\Psi(g_1K\, g_2K) = \Psi(g_1g_2K) = \varphi(g_1g_2) = \varphi(g_1)\varphi(g_2) = \Psi(g_1K)\,\Psi(g_2K)$

Claim 3: Ψ is injective

Pf/ $\Psi(gK) = e' \iff \varphi(g) = e' \iff g \in K \iff gK = K$.

Claim 4: $\overline{\varphi} = \Psi$ with range restricted to $H \doteq \operatorname{Im}(\varphi)$

By definition $\overline{\varphi}$ is surjective & injective, so it is a bijection.

Exercise: Bijective group homomorphisms are isomorphisms (HW1). $\square$

The other two isomorphism theorems will follow from this one.

$$\boxed{\text{First Iso Thm: } \quad \frac{G}{\ker\varphi} \xrightarrow{\ \sim\ } \operatorname{Im}\varphi}$$

§2 Second Isomorphism Thm:

Thm 2: Let G be a group and $N \lhd G$ a normal subgp. Then

(i) The assignment $H \longmapsto H/N$ is a bijection between

$$\left\{\begin{array}{l}\text{Subgroups of}\\ G \text{ containing } N\end{array}\right\} \longleftrightarrow \{\text{Subgroups of } G/N\}$$

(ii) Let $H < G$ be a subgroup containing N. Then

H is normal if and only if H/N is normal in G/N

Furthermore, we have $\dfrac{G}{H} \xrightarrow{\ \sim\ } \dfrac{G/N}{H/N}$ $\quad gH \longmapsto gH/N$

Proof of (i) Let $\pi: G \longrightarrow G/N$ be the natural surjection. & pick $H<G$

Claim 1 If $N \subseteq H$, then $\pi(H) = \{hN : h \in H\} < G/N$

Pf/ $\bullet\ e_{G/N}$ = identity of G/N $= eN \in \pi(H)$ ✓

- $(h_1 N)(h_2 N) = h_1 h_2 N$ $\forall h_1, h_2 \in H$, so law of composition holds for $\pi(H)$
- $(hN)^{-1} = h^{-1} N$ $\forall h \in H$, so $\pi(H)$ is closed under inverses. □

For the converse, pick $\overline{H} < G/N$ a subgroup, let $H = \pi^{-1}(\overline{H})$.

<u>Claim 2</u>: $H < G$ is a subgroup of G containing N & $\pi(H) = \overline{H}$.

Pf/ $N = \pi^{-1}(\{e_{G/N}\}) = \operatorname{Ker}\pi \subset \pi^{-1}(\overline{H}) \overset{\text{def}}{=} H$.

By def: $H = \{g \in G : \pi(g) \in \overline{H}\}$ <u>Want to show:</u> $H < G$

- $e \in H$ is clear since $e \in N \subset H$ ✓
- $g_1, g_2 \in H \Rightarrow \pi(g_1 g_2) = \pi(g_1)\pi(g_2) \in \overline{H} \cdot \overline{H} = \overline{H}$.
  so $g_1 g_2 \in H$. ✓
- $g \in H \Rightarrow \pi(g^{-1}) = \pi(g)^{-1} \in \overline{H}^{-1} = \overline{H} \Rightarrow g^{-1} \in H$ ✓

Finally: $\pi(H) = \overline{H}$ by the surjectivity of π □

<u>Proof of (ii)</u>: Fix H subgroup of G with $N \subset H$ & set $\boxed{\overline{H} := \pi(H) = H/N}$

$$H \lhd G \iff ghg^{-1} \in H \ \forall g \in G \ \forall h \in H \underset{\text{def } \overline{H},\, N \subseteq H}{\iff} ghg^{-1}N \in \overline{H} \ \forall g \in G \ \forall h \in H$$

$$\iff (gN)(hN)(g^{-1}N) \in \overline{H} \ \forall g \in G \ \forall h \in H \iff \overline{H} \lhd G/N$$

. To finish, assume $N \subset H$ & $N \lhd G$, $H \lhd G$. Then

$$\begin{tikzcd} G \arrow[r, two heads, "\pi_1"] \arrow[rr, bend right, "\psi = \pi_1 \circ \pi_2"'] & G/N \arrow[r, two heads, "\pi_2"] & (G/N)/(H/N) \end{tikzcd}$$

- ψ is composition of gp homomorphisms, so it is also a gp hom.
- ψ is surjective (comp of surjections)
- <u>$\operatorname{Ker}\psi = H$</u>: $\psi(g) = e \iff \pi_1(g) \in H/N \iff gN \in H/N$

$\Longleftrightarrow \; gN = hN$ for some h $\Longleftrightarrow \; g \in hN \underset{\substack{\downarrow \\ N \subseteq H}}{\subseteq} H$.

Now, by 1^{st} Isomorphism Thm: $\quad G/H \underset{\Psi}{\cong} {}^{G/N}\big/_{H/N}$. $\square$

- Asides from the proofs of both (i) & (ii):

Prop 1: For any group homomorphism $\varphi : G_1 \longrightarrow G_2$, if $H_1 < G_1$ is a subgp then $\varphi(H_1) < G_2$ is a subgroup.

Prop 2: For any group homomorphism $\varphi : G_1 \longrightarrow G_2$ & $N_2 \triangleleft G_2$, then $\varphi^{-1}(N_2) \triangleleft G_1$.

§3. Third Isomorphism Thm:

Thm 3: Let G be a group, $H < G$ a subgroup & $N \triangleleft G$. Then:

(i) $H \cap N \subset H$ is a normal subgroup

(ii) $HN := \{hx : h \in H, x \in N\} \subset G$. Then $HN = NH$ & HN is a subgroup of G.

(iii) $N \subset HN$ is a normal subgroup.

(iv) $$\begin{array}{ccc} H/H\cap N & \longrightarrow & HN/N \\ h(H\cap N) & \longmapsto & hN \end{array}$$ is an isomorphism

$\boxed{\text{Cartoon encoding (i) — (iii)}}$

$H < G$
$N \triangleleft G$ $\rightsquigarrow$

$$\begin{tikzcd} & G \arrow[d, dash, "\vee"] & \\ & HN = NH \arrow[dl, dash, "\vee"'] \arrow[dr, dash, "\triangleright"] & \\ H \arrow[dr, dash, "\triangleright"'] & & N \arrow[dl, dash, "<"] \\ & H \cap N & \end{tikzcd}$$

Proof of (i): Want to show $H\cap N \trianglelefteq H$. Pick $h\in H$ & $x\in H\cap N$.

Then: $\left.\begin{array}{l} \bullet\ hxh^{-1}\in H \text{ because } H<G \\ \bullet\ hxh^{-1}\in N \text{ — } N\trianglelefteq G \end{array}\right\} \Rightarrow hxh^{-1}\in H\cap N$ □.

Proof of (ii): We first show $HN\subseteq NH$. Pick $hx\in HN$ ($h\in H$, $x\in N$)

Claim 1: $hx\in NH$ (Pf/ $hx = \underbrace{hxh^{-1}}_{\in N\trianglelefteq G}h \in NH$ ✓)

- Proof of $NH\subset HN$ is similar. ($xh = h\underbrace{h^{-1}xh}_{\in N}\in HN$)
- **Claim 2:** HN is a subgroup of G

Pf/ • $e=e.e\in HN$ ✓

- $(h_1x_1)(h_2x_2) = h_1h_2\underbrace{h_2^{-1}x_1h_2}_{\in N\ (N\trianglelefteq G)}x_2 \in HN$
- $(hx)^{-1} = x^{-1}h^{-1}\in NH = HN$ by Claim 1 for all $h\in H$, $x\in N$. □

Proof of (iii): $N\trianglelefteq G$ & $N=eN < HN \subset G$ $\Rightarrow$ $N\trianglelefteq HN$.

Proof of (iv): Consider the composition of group homomorphisms

$$\begin{tikzcd} H \arrow[r, hook, "i", "\text{inclusion}"'] \arrow[rr, bend left, "\varphi = \pi\circ i"] & HN \arrow[r, two heads, "\pi", "\text{projection}"'] & HN/N \end{tikzcd}$$

- φ is a group homomorphism

Claim 1: φ is surjective

Pf/ $\left.\begin{array}{l} hxN = hN \quad \text{for } h\in H,\ x\in N \\ \text{But } hN = i(h) \end{array}\right\} \varphi(h) = hxN.$

Claim 2 $\operatorname{Ker}\varphi = H\cap N$

Pf/ $h\in \operatorname{Ker}\varphi$ ($h\in H$) $\Leftrightarrow$ $\varphi(h)=\bar{e}$ (= identity of HN/N) (& $h\in H$)

$\Leftrightarrow$ $hN = N$ ($h\in H$) $\Leftrightarrow$ $h\in H\cap N$

Then, by First Isomorphism Thm, we get $\dfrac{H}{H\cap N}\simeq\dfrac{HN}{N}$