

# Lecture IV: Basic Isomorphism Theorems

Last time: Defined (normal) subgroups generated by a set.

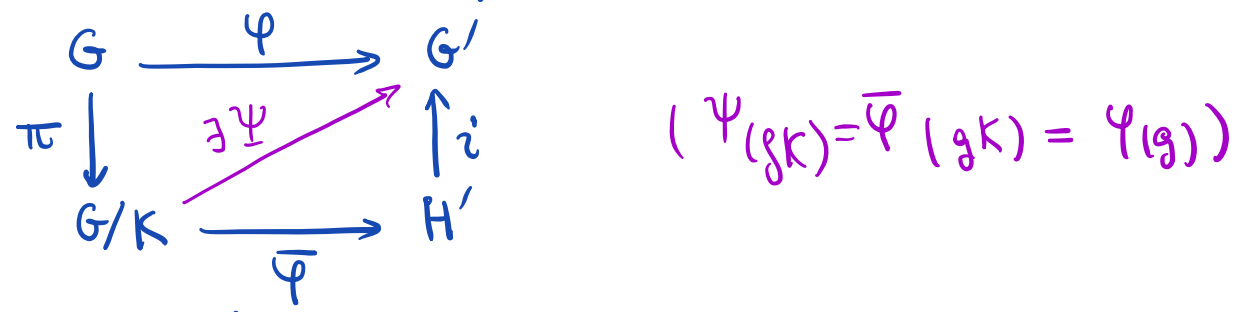
- Left cosets  $G/H = \{xH : x \in G\} / \sim$       $x \sim y \Leftrightarrow x^{-1}y \in H$
- Right —  $H/G = \{Hx : x \in G\} / \sim$       $x \sim y \Leftrightarrow xy^{-1} \in H$
- Thm: If  $H \triangleleft G$ , then  $G/H$  is a group under  $gH * g'H = gg'H$   
&  $G \twoheadrightarrow G/H$  is sp hom with  $\text{Ker } G = H$ .
- Cyclic groups & their classification  $\begin{cases} G \text{ infinite} \cong \mathbb{Z} \\ G \text{ finite} \cong \mathbb{Z}/n\mathbb{Z} \end{cases}$   
( $n = |G|$ )
- first counting lemma.

TODAY, Discuss 3 Isomorphisms in Group Theory.

## § 1 First Isomorphism Theorem:

Theorem 1: Let  $G, G'$  be two groups and  $\varphi: G \rightarrow G'$  be a group homomorphism. Write  $K = \text{Ker}(\varphi) \triangleleft G$  &  $H' = \text{Im}(\varphi) \triangleleft G'$ .

Then we have a commutative diagram:



Here:  $\pi$  = natural projection &  $i$  is a natural inclusion.

Moreover,  $\bar{\varphi}$  is an isomorphism

Proof: Define  $\psi: G/K \rightarrow G'$  by  $\psi(gk) = \varphi(g)$

Claim 1:  $\psi$  is well-defined ( $g_1k = g_2k \stackrel{?}{\Rightarrow} \varphi(g_1) = \varphi(g_2)$ )

Bf:  $g_1k = g_2k \Leftrightarrow g_2^{-1}g_1 \in K$ , so  $\varphi(g_2^{-1}g_1) = \varphi(g_2)^{-1}\varphi(g_1) = e'$

Thus:  $\varphi(g_1) = \varphi(g_2) \checkmark$

Claim 2:  $\Psi$  is a group homomorphism:

Pf/  $\Psi(g_1 k g_2 k) = \Psi(g_1 g_2 k) = \Psi(g_1 g_2) = \Psi(g_1) \Psi(g_2) = \Psi(g_1 k) \Psi(g_2 k)$

Claim 3:  $\Psi$  is injective

Pf/  $\Psi(gk) = e' \iff \Psi(g) = e' \iff g \in K \iff gk = k.$

Claim 4:  $\bar{\Psi} = \Psi$  with range restricted to  $H' = \text{Im}(\Psi)$

By definition  $\bar{\Psi}$  is surjective & injection, so it is a bijection.

Exercise: Bijective group homomorphisms are isomorphisms (HW1).  $\square$

The other two isomorphism theorems will follow from this one.

First Iso Thm:  $\frac{G}{\text{ker } \Psi} \xrightarrow{\sim} \text{Im } \Psi$

§2 Second Isomorphism Thm:

Thm 2: Let  $G$  be a group and  $N \triangleleft G$  a normal subgp. Then

(i) The assignment  $H \mapsto H/N$  is a bijection between

$$\left\{ \begin{array}{l} \text{Subgroups of } \\ G \text{ containing } N \end{array} \right\} \longleftrightarrow \left\{ \text{Subgroups of } G/N \right\}$$

(ii) Let  $H < G$  be a subgroup containing  $N$ . Then

$H$  is normal if and only if  $H/N$  is normal in  $G/N$

Furthermore, we have  $\frac{G}{H} \xrightarrow{\sim} \frac{G/N}{H/N} \quad gH \mapsto gH/N$

Proof of (i) Let  $\pi: G \rightarrow G/N$  be the natural surjection. & pick  $H < G$

Claim 1 If  $N \subseteq H$ , then  $\pi(H) = \{ hN : h \in H \} < G/N$

Pf •  $e_{G/N} = \text{identity of } G/N = eN \in \pi(H) \checkmark$

- $(h_1 N)(h_2 N) = h_1 h_2 N \quad \forall h_1, h_2 \in H$ , so law of composition holds for  $\pi(H)$
- $(hN)^{-1} = h^{-1}N \quad \forall h \in H$ , so  $\pi(H)$  is closed under inverses.  $\square$

For the converse, pick  $\bar{H} < G/N$  a subgroup, let  $H = \pi^{-1}(\bar{H})$ .

Claim 2:  $H < G$  is a subgroup of  $G$  containing  $N$  &  $\pi(H) = \bar{H}$ .

Pf:  $N = \pi^{-1}(\{e_{G/N}\}) = \text{Ker } \pi \subset \pi^{-1}(\bar{H}) = H$ .

By def:  $H = \{g \in G : \pi(g) \in \bar{H}\}$  Want to show:  $H < G$

- $e \in H$  is clear since  $e \in N \subset H$  ✓
- $g_1, g_2 \in H \Rightarrow \pi(g_1 g_2) = \pi(g_1) \pi(g_2) \in \bar{H} \cdot \bar{H} = \bar{H}$ .  
so  $g_1 g_2 \in H$ . ✓
- $g \in H \Rightarrow \pi(g^{-1}) = \pi(g)^{-1} \in \bar{H}^{-1} = \bar{H} \Rightarrow g^{-1} \in H$  ✓

Finally:  $\pi(H) = \bar{H}$  by the surjectivity of  $\pi$   $\square$

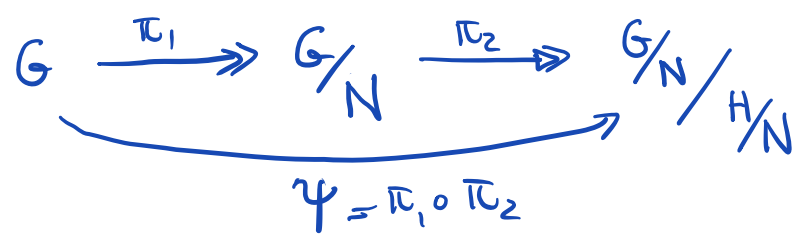
Proof of (ii): Fix  $H$  subgroup of  $G$  with  $N \subset H$  & set  $\bar{H} := \pi(H) = H/N$

$$H \triangleleft G \iff ghg^{-1} \in H \quad \forall g \in G \quad \forall h \in H \iff ghg^{-1}N \in \bar{H} \quad \forall g \in G \quad \forall h \in H$$

$\text{def } \bar{H}, N \subseteq H$

$$\iff (gN)(hN)(g^{-1}N) \in \bar{H} \quad \forall g \in G \quad \forall h \in H \iff \bar{H} \triangleleft G/N$$

To finish, assume  $N \subset H$  &  $N \triangleleft G$ ,  $H \triangleleft G$ . Then



- $\psi$  is composition of gp homomorphisms, so it is also a gp hom.
- $\psi$  is surjective (comp of surjections)
- Ker  $\psi = H$ :  $\psi(g) = e \iff \pi_1(g) \in H/N \iff gN \in H/N$

$$\Leftrightarrow gN = hN \text{ for some } h \Leftrightarrow g \in hN \subseteq H.$$

$\downarrow$   
 $N \subseteq H$

Now, by 1<sup>st</sup> Isomorphism Thm:  $G/H \cong G/N/H/N$ . □

• Besides from the proofs of both (i) & (ii):

Prop 1: For any group homomorphism  $\varphi: G_1 \rightarrow G_2$ , if  $H_1 < G_1$  is a subgroup then  $\varphi(H_1) < G_2$  is a subgroup.

Prop 2: For any group homomorphism  $\varphi: G_1 \rightarrow G_2$  &  $N_2 \triangleleft G_2$ , then  $\varphi^{-1}(N_2) \triangleleft G_1$ .

§ 3. Third Isomorphism Thm:

Thm 3: Let  $G$  be a group,  $H < G$  a subgroup &  $N \triangleleft G$ . Then:

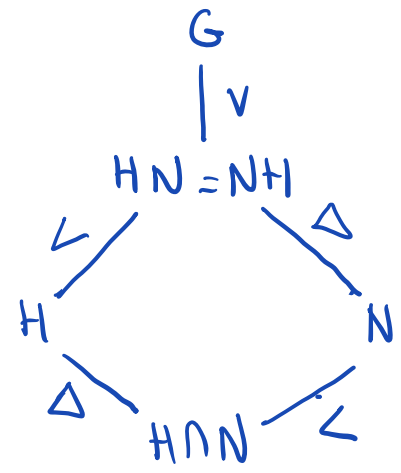
- (i)  $H \cap N < H$  is a normal subgroup
- (ii)  $HN := \{hx : h \in H, x \in N\} \subset G$ . Then  $HN = NH$  &  $HN$  is a subgroup of  $G$ .
- (iii)  $N < HN$  is a normal subgroup.

(iv)  $\frac{H}{H \cap N} \longrightarrow \frac{HN}{N}$  is an isomorphism

$h(H \cap N) \longmapsto hN$

Cartoon encoding (i) - (iii)

$H < G$   
 $N \triangleleft G$



Proof of (i): Want to show  $H \cap N \triangleleft H$ . Pick  $h \in H$  &  $x \in H \cap N$

Then:  $\left. \begin{array}{l} \bullet h x h^{-1} \in H \text{ because } H < G \\ \bullet h x h^{-1} \in N \text{ — } N \triangleleft G \end{array} \right\} \Rightarrow h x h^{-1} \in H \cap N \quad \square$

Proof of (ii): We first show  $HN \subseteq NH$ . Pick  $hx \in HN$  ( $h \in H, x \in N$ )

Claim 1  $hx \in NH$  (Pf/  $hx = \underbrace{hxh^{-1}}_{\in N} h \in NH \checkmark$ )

• Proof of  $NH \subset HN$  is similar. ( $xh = \underbrace{hh^{-1}xh}_{\in N} \in HN$ )

• Claim 2:  $HN$  is a subgroup of  $G$

Pf/.  $e = e \cdot e \in HN \checkmark$

•  $(h_1 x_1)(h_2 x_2) = h_1 h_2 \underbrace{h_2^{-1} x_1 h_2}_{\in N} x_2 \in HN$   
( $N \triangleleft G$ )

•  $(hx)^{-1} = x^{-1}h^{-1} \in NH = HN$  by Claim 1 for all  $h \in H, x \in N$ .  $\square$

Proof of (iii):  $N \triangleleft G$  &  $N = eN < HN \subset G \Rightarrow N \triangleleft HN$ .

Proof of (iv): Consider the composition of group homomorphisms

$$H \xrightarrow[\text{inclusion}]{i} HN \xrightarrow[\text{projection}]{\pi} HN/N$$

$\varphi = \pi \circ i$

•  $\varphi$  is a group homomorphism

Claim 1:  $\varphi$  is surjective

Pf/  $hxN = hN$  for  $h \in H, x \in N$  }  $\varphi(h) = hxN$ .  
But  $hN = i(h)$

Claim 2  $\text{Ker } \varphi = H \cap N$

Pf/  $h \in \text{Ker } \varphi \Leftrightarrow \varphi(h) = \bar{e}$  (= identity of  $HN/N$ ) ( $h \in H$ )  
( $h \in H$ )  $\Leftrightarrow hN = N$  ( $n \in H$ )  $\Leftrightarrow h \in H \cap N$

Then, by First Isomorphism Thm, we get  $\frac{H}{H \cap N} \cong \frac{HN}{N}$

