Lecture VI: Group Actions on Sets

So far: (i) Defined useful terms from Group Theory:
- Group, subgroup, subgroup generated by a subset, order & exponent
- Normal subgroup, normal subgp
- Left/Right cosets (\(G/H\) & \(H\backslash G\)), Quotient groups
- Group homomorphisms / Isomorphisms, Kernel & Image
- Free group, Generators & Relations; Examples (Free(\(\mathbb{N}\)), \(S_n\), \(D_n\))

(ii) Main Results: 3 Isomorphism Theorems, Classification of cyclic gps.

TODAY: Groups acting on sets

3.1. Group actions:

Def: Let \(G\) be any group and let \(X\) be a set. A (left) action of \(G\) on \(X\) is a set map
\[
G \times X \to X
\]
\[
(g, x) \mapsto \alpha(g, x) = g \cdot x
\]

(i) \(e \cdot x = x\) \(\forall x \in X\)

(ii) \((g_1g_2) \cdot x = g_1 \cdot (g_2 \cdot x)\) \(\forall g_1, g_2 \in G\) and \(x \in X\).

[For a right action we replace (i) by (ii) \((x \cdot g_1)g_2 = x \cdot (g_1g_2)\) ]

Observation: If \(G \leq X\), then each \(g \in G\) defines a set map:

\[
\mathcal{L}(g) = \mathcal{L}(g, -): X \to X
\]

\[
x \mapsto g \cdot x
\]

It satisfies:

(i) \(\mathcal{L}(e) = \text{Id}_X\)

(ii) \(\mathcal{L}(g_1g_2)(x) = (g_1g_2)x = g_1 \cdot (g_2 \cdot x) = \mathcal{L}(g_1) \circ \mathcal{L}(g_2)(x) = (\mathcal{L}(g_1) \circ \mathcal{L}(g_2))(x)\)

(iii) \(\mathcal{L}(g_1^{-1}) \circ \mathcal{L}(g_1) = \mathcal{L}(g_1^{-1}g_1) = \mathcal{L}(e) = \text{Id}_X\) by (i)

Conclusion: \(\mathcal{L}: G \to \text{Aut}(X) = \{ f: X \to X \text{ bijection} \} \) is a group homomorphism.

Set = Symmetric gp on \(X\)
Example \( G = \text{GL}_n(\mathbb{R}) \) \( \subset \) \( \mathbb{R}^n \) by
\[
G \times X \xrightarrow{\times} X \quad \text{matrix multiplication}
\]
\[
(A, x) \mapsto Ax
\]
Want to highlight that for all \( A \in \text{GL}_n(\mathbb{R}) \), the resulting map \( \mathbb{R}^n \to \mathbb{R}^n \) is not just a set bijection, but it preserves the vector space structure:
\[
A(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) = \alpha A(\mathbf{v}_1) + \beta A(\mathbf{v}_2) \quad \forall \alpha, \beta \in \mathbb{R}
\]
Thus, \( \text{GL}_n(\mathbb{R}) = \text{Aut}_{\mathbb{R}^n}(\mathbb{R}^n) \).

5.2. Orbits, Stabilizers & Fixed Points

Fix \( G \subset X \).

**Def.** The **orbit** of an element \( x \in X \) is the following subset of \( X \):
\[
G \cdot x := \{ g \cdot x \mid x \in G \} \subseteq X
\]

**Def.** The **stabilizer** of an element \( x \in X \) is the following subgroup of \( G \):
\[
\text{Stab}_G(x) := \{ g \in G \mid g \cdot x = x \} \subseteq G
\]

**Obs.** \( \text{Stab}_G \) need not be a normal subgroup (Example on page 5).

**Def.** The **fixed point** set of an element \( g \in G \) is the following subset of \( X \):
\[
X^g := \{ x \in X \mid g \cdot x = x \} \subseteq X
\]

**Example:** \( D_n \xrightarrow{\cong} \text{GL}_2(\mathbb{R}) = \text{Aut}(\mathbb{R}^2) \)
\[
s \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{(reflection about x-axis)}
\]
\[
p \mapsto \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad \text{(rotation of angle } \theta = \frac{2\pi}{n})
\]

- This defines a unique group homomorphism because \( \mathbb{Z}(s^2) = \mathbb{Z}(p^n) = \mathbb{Z}((s^2)p^n) = \text{Id}_2 \).
Since \( b \) is a group homomorphism, it defines an action \( D_n \times \mathbb{R}^2 \). Also on \( X = \mathbb{R}^2 \setminus \{(0)\} \). [\( D_n \) fixes \( (0) \), so we can ignore it]

Let us compute the orbit of a pt \( p \in X \).

In particular, \( \{ p, \theta(p), \ldots, \theta^{n-1}(p) \} = \mathbb{R} \). (x)

\[
D_n \cdot p = \{ p, \theta(p), \theta^2(p), \ldots, \theta^{n-1}(p), s(p), s \theta(p), \ldots, s \theta^{n-1}(p) \}
\]

Claim: \( |D_n \cdot p| = 2n \iff s(p) \notin \{ p, \theta(p), \ldots, \theta^{n-1}(p) \} \)

(\(\Rightarrow\)) If \( s(p) = \theta^r(p) \) for some \( r = 0, \ldots, n-1 \), we have a repeated element .

(\(\Leftarrow\)) If \( |D_n \cdot p| < 2n \), then there is a repeated element \( x \).

Now \( x = s \theta^k(p) \) or \( \theta^j(p) \) for some \( k, j \).

But \( s \theta^k(p) = s \theta^j(p) \) and \( \theta^k(p) \neq \theta^j(p) \) by (x). So the only option is \( x = s \theta^k(p) = \theta^j(p) \) for some \( k, j \).

But \( s \theta^k = \theta^{k+1} = \theta^{n-k} s \) \( \Rightarrow \theta^{k+1} s \theta^k = \theta^{n-k} s \theta^k \)

\( s(p) = \theta^{k+j}(p) \) \( \Leftarrow \)

Now: \( s(p) = \theta^r(p) \) for some \( r = 0, \ldots, n-1 \) means

Reflecting \( p \) about \( x \)-axis is rotation by \( \frac{2 \pi r}{n} \).

In particular \( \theta^r(p) = p \).

Write \( p = Re^{\beta i} \) for \( \beta = \frac{2 \pi r}{n} \).

Then:

\[
Re^{\beta i} = Re^{-(\beta + \frac{2 \pi r}{n})i} \iff \beta = -\frac{2 \pi r}{n} \text{ (215)}
\]

\[
\beta = -\frac{15 \pi}{n} \text{ (15)}
\]
$|D_n \cdot p| < 2n \iff p = Re^{R \beta}$ with $\beta = \frac{n - r}{n}$ or

$\beta = \frac{2n - r}{n}$

($0 \leq r < n$)

In short $p = R \left[ \cos \left( \frac{n - r}{n} \theta \right) \right] = R \left[ \frac{\cos \left( \frac{r}{n} \theta \right)}{\sin \left( \frac{n - r}{n} \theta \right)} \right]$ or $p = R \left[ \cos \left( \frac{2n - r}{n} \theta \right) \right] = R \left[ -\frac{\cos \left( \frac{r}{n} \theta \right)}{\sin \left( \frac{2n - r}{n} \theta \right)} \right]$

In this case, $|D_n \cdot p| = n$, since $\{p, p(p), \ldots, p^{n-1}(p)\} = D_n \cdot p$

Q: $\text{Stab}_{D_n}(p) =$ ?

A: $<sp^r> \leq \text{Stab}_{D_n}(p)$. We'll see next that equality holds since $|\text{Stab}_{D_n}(p)| = \frac{|D_n|}{|D_n \cdot p|} = \frac{2n}{n} = 2$.

Alternative: $p) \in \text{Stab}_{D_n}(p)$ under $j = 0$. By the order description:

$p^j(p) = p = sp^r(p) \iff p^j = sp^r(p) \iff j = r$

Conclusion: $\text{Stab}_{D_n}(p) = \{s, p^r, e\} = <sp^r>$

Proposition: Let $G \subseteq X$

(1) For every $x \in X$ we have a (set) bijection

$G / \text{Stab}_G(x) \longrightarrow G \cdot x$

(2) For every $\sigma \in G$ and $x \in X$, we have an isomorphism of groups

$\text{Conjugation by } \sigma: \text{Stab}_G(x) \longrightarrow \text{Stab}_G(\sigma \cdot x)$

Proof: (1) Define $\alpha(x): G \longrightarrow G \cdot x$, surjective by definition.
But \( g \cdot x = h \cdot x \iff h^{-1}g \cdot x = x \iff h^{-1}g \in \text{Stab}_G(x) \). So this map factors through \( G/\text{Stab}_G(x) \) is a set!

\[
\begin{array}{ccc}
G & \xrightarrow{f} & G \cdot x \\
\downarrow \circ \sim & & \Downarrow \tilde{f} \\
G/\text{Stab}_G(x) & & \\
\end{array}
\]

\( \tilde{f} \) is a bijection.

(2) \( g \in \text{Stab}_G(x) \iff g \cdot x = x \iff (\sigma g \sigma^{-1})x = \sigma x \)
\[ \iff \sigma g \sigma^{-1} \in \text{Stab}_G(\sigma x) \]

Since \( \text{Cmj}_G \) is an automorphism of \( G \), it induces an isomorphism between \( \text{Stab}_G(x) \) and \( \text{Stab}_G(\sigma x) \).

\[ \text{§ 3 Counting Lemmas} \]

\[ \text{Def: } x \sim_G x' \text{ in } X \text{ iff } \exists g \in G, g \cdot x = x' \]

\[ \text{Claim: This defines an equivalence relation.} \]

\[ G\backslash X := X/\sim_G \text{ equiv classes } = \text{orbits} \]

\[ \text{Easy Observation: } X = \bigsqcup_{x \in G\backslash X} G \cdot x_x \Rightarrow |X| = \sum_{x \in G\backslash X} |G \cdot x_x| \]

(Here \( x_x \in X \) is a choice of an element from the \( G \)-orbit labeled by \( x \in G \).)

Recall \( G/\text{Stab}_G(x) \) \( \xrightarrow{\text{bij}} \) \( G \cdot x \) \& \( \text{Stab}_G(x) \) \( \xrightarrow{\text{conj}} \) \( \text{Stab}_G(\sigma x) \)

\[ \text{Corollary: } \]

(1) \( |G| = |G \cdot x| \cdot |\text{Stab}_G(X) \cdot x| \forall x \in X \)

(2) \( |X| = \sum_{x \in G\backslash X} |G|/|\text{Stab}_G(x_x)| \)

Next time: Burnside's Lemma for counting orbits.