Lecture VI: Group Actions on Sets

So far: (1) Defined useful terms from Group Theory:

- Group, subgroup, subgroup generated by a subset, order & exponent
- Normal subgroup, normal subgp ————
- Left / Right cosets (G/H & $H\backslash G$), Quotient groups
- Group homomorphisms / Isomorphisms, Kernel & Image of gp hom.
- Free group, Generators & relations; Examples ($\mathrm{Free}(A)$, S_n, D_n)

(2) Main Results: 3 Isomorphism Thms, Classification of cyclic gps.

TODAY: Groups acting on sets

§1. Group actions:

Def: Let G be any group and let X be a set. A (left) action of G on X is a set map

$$\begin{aligned} G \times X &\xrightarrow{\alpha} X \\ (g, x) &\longmapsto \alpha(g,x) =: g\cdot x \end{aligned}$$

satisfying

NOTATION $G \circlearrowright X$

(i) $e\cdot x = x \quad \forall x \in X$

(ii) $(g_1 g_2)\cdot x = g_1\cdot(g_2\cdot x) \quad \forall g_1, g_2 \in G$ and $x \in X$.

[For a right action we replace (ii) by (ii') $(x\cdot g_1)\cdot g_2 = x\cdot(g_1 g_2)$]

$X \circlearrowleft G$

Observation: If $G \circlearrowright X$, then each $g \in G$ defines a set map:

$$\begin{aligned} \tau(g) := \alpha(g, -) : X &\longrightarrow X \\ x &\longmapsto g\cdot x \end{aligned}$$

It satisfies:

(i) $\tau(e) = \mathrm{Id}_X$

(ii) $\tau(g_1 g_2)(x) = (g_1 g_2)x = g_1\cdot(g_2\cdot x) = \tau(g_1)(\tau(g_2)(x)) = (\tau(g_1)\circ\tau(g_2))(x)$

(iii) $\tau(g_1^{-1})\circ\tau(g_1) = \tau(g_1^{-1}g_1) = \tau(e) = \mathrm{Id}_X$ by (i)

Conclusion: $\tau : G \longrightarrow \mathrm{Aut}_{\mathrm{Set}}(X) := \{f : X \to X \text{ bijection}\}$ is a gp homomorphism.

$= \text{Symmetric gp on } X$

<u>Example</u> $G = GL_n(\mathbb{R}) \circlearrowright X = \mathbb{R}^n$ by

$$G \times X \xrightarrow{\alpha} X \qquad \text{matrix multiplication.}$$
$$(A, \underline{x}) \longmapsto A\underline{x}$$

Want to highlight that for all $A \in GL_n(\mathbb{R})$ the resulting map $\mathbb{R}^n \xrightarrow{A} \mathbb{R}^n$ is not just a set bijection, but it <u>preserves the vector space structure</u> $(A(\alpha v_1 + \beta v_2) = \alpha A(v_1) + \beta A(v_2)$ $\forall \alpha, \beta \in \mathbb{R}$ $\forall v_1, v_2 \in \mathbb{R}^n)$

Thus, $GL_n(\mathbb{R}) = \mathrm{Aut}_{\mathbb{R}\text{-v.s.}}(\mathbb{R}^n)$.

§2 Orbits, Stabilizers & Fixed Points

Fix $G \circlearrowright X$.

<u>Def</u>: The <u>orbit</u> of an element $x \in X$ is the following subset of X

$$\boxed{G \cdot x} := \{ g \cdot x \mid x \in G \} \subseteq X$$

<u>Def</u> The <u>stabilizer</u> of an element $x \in X$ is the following subgroup of G

$$\boxed{\mathrm{Stab}_G(x)} := \{ g \in G \mid g \cdot x = x \} \leq G$$

<u>Obs</u>: Stab_G need <u>not</u> be a normal subgroup (Example on page 5)

<u>Def</u>: The <u>fixed point set</u> of an element $g \in G$ is the following subset of X:

$$\boxed{X^g} := \{ x \in X \mid g \cdot x = x \} \subseteq X$$

$\langle s, \rho \mid s^2, \rho^n, (s\rho)^2 \rangle$

<u>Example</u>: $D_n \overset{\tau}{\hookrightarrow} GL_2(\mathbb{R}) = \mathrm{Aut}(\mathbb{R}^2)$

$$s \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{(reflection about x-axis)}$$

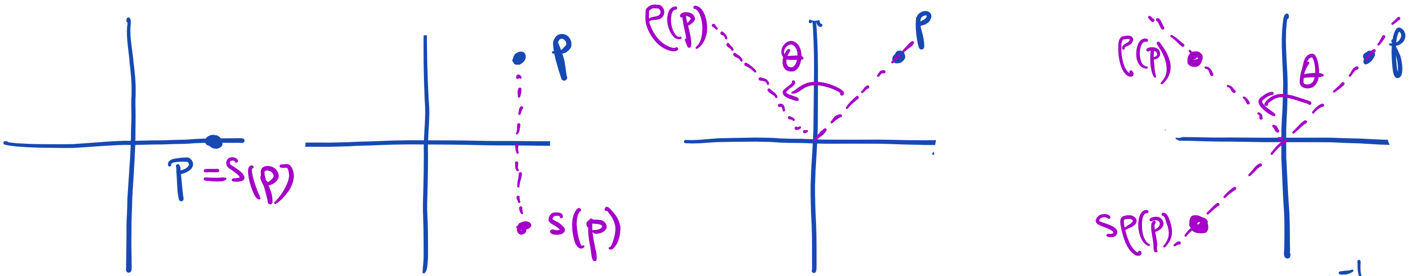
$$\rho \longmapsto \begin{bmatrix} \cos(\theta) & -\sin\theta \\ \sin(\theta) & \cos\theta \end{bmatrix} \quad \left(\text{rotation of angle } \theta = \frac{2\pi}{n}\right)$$

- This defines a unique group homomorphism because $\tau(s^2) = \tau(\rho^n) = \tau((s\rho)^2) = Id_2$.

- Since τ is a group homomorphism, it defines an action $D_n \curvearrowright \mathbb{R}^2$.

Also on $X = \mathbb{R}^2 \setminus \{\binom{0}{0}\}$. [$D_n$ fixes $\binom{0}{0}$, so we can ignore it]

- Let us compute the orbit of a pt $p \in X$.



$s\rho = \rho^{-1}s$

In particular, $\boxed{|\{p, \rho(p), \ldots, \rho^{n-1}(p)\}| = n}$ $(*)$

$D_n \cdot p = \{p, \rho(p), \rho^2(p), \ldots\ldots, \rho^{n-1}(p), s_{(p)}, s\rho_{(p)}, \ldots, s\rho^{n-1}_{(p)}\}$

$\supseteq \{p, \rho(p), \ldots\ldots \rho^{n-1}(p)\}$

<u>Claim</u>: $|D_n \cdot p| = 2n \iff s(p) \notin \{p, \rho(p), \ldots, \rho^{n-1}(p)\}$

Pf/ $(\Longrightarrow)$ If $s(p) = \rho^r(p)$ for some $r = 0, \ldots, n-1$, we have a repeated element
<u>Contr</u>!

$(\Longleftarrow)$ If $|D_n \cdot p| < 2n$, then there is a repeated element x.

Now $x = s\rho^k(p)$ or $\rho^j(p)$ for some k, j

BUT $s\rho^k(p) \neq s\rho^j(p)$ & $\rho^k(p) \neq \rho^j(p)$ by $(*)$. So the only option is $x = s\rho^k(p) = \rho^j(p)$ for some k, j

BUT $s\rho^k = \rho^{-k}s = \rho^{n-k}s \implies \rho^{n-k}s(p) = \rho^j(p)$

$s(p) = \rho^{k+j}(p)$ Contr!

$\square$

<u>Now</u>: $s(p) = \rho^r(p)$ for some $r = 0, \ldots, n-1$ means

Reflecting p about x-axis = rotation by $\frac{2\pi r}{n}$.

In particular $s\rho^r(p) = p$

Write $p = Re^{\beta i}$ for $0 \leq \beta < 2\pi$ Then:

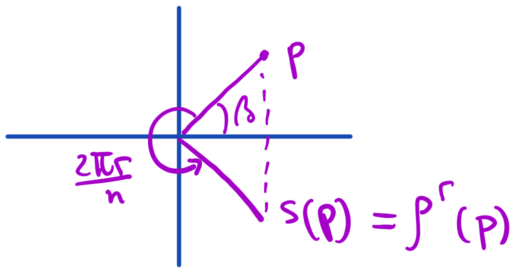
$Re^{\beta i} = Re^{-(\beta + \frac{2\pi r}{n})i} \iff \beta \equiv -\beta - \frac{2\pi r}{n} \ (2\pi)$

$\beta = \frac{-\pi r}{n} \ (\pi)$.

$|D_n \cdot p| < 2n \iff p = Re^{\beta i}$ with $\beta = \frac{n-r}{n}\pi$ or

$\beta = \frac{2n-r}{n}\pi$

$(0 \le r < n)$



In short $p = R\begin{bmatrix} \cos\left(\frac{n-r}{n}\pi\right) \\ \sin\left(\frac{n-r}{n}\pi\right) \end{bmatrix} = R\begin{bmatrix} \sin\frac{r\pi}{n} \\ \cos\frac{r\pi}{n} \end{bmatrix}$ or $p = R\begin{bmatrix} \cos\frac{\pi r}{n} \\ -\sin\frac{\pi r}{n} \end{bmatrix}$

In this case, $|D_n \cdot p| = n$. since $\{p, \rho(p), \dots, \rho^{n-1}(p)\} = D_n \cdot p$.

Q: $\mathrm{Stab}_{D_n}(p) = ?$

A $\langle s\rho^r \rangle \subseteq \mathrm{Stab}_{D_n}(p)$. We'll see next that equality holds since $|\mathrm{Stab}_{D_n}(p)| = \frac{|D_n|}{|D_n \cdot p|} = \frac{2n}{n} = 2$.

Alternative: $\rho^j \notin \mathrm{Stab}_{D_n}(p)$ unless $j = 0$. by the order description.

$s\rho^j(p) = p = s\rho^r(p) \iff \rho^j(p) = \rho^r(p) \iff j = r$

rotations only fix $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Conclusion: $\mathrm{Stab}_{D_n}(p) = \{s\rho^r, e\} = \langle s\rho^r \rangle$ $\square$

Proposition: Let $G \curvearrowright X$

(1) For every $x \in X$ we have a (set) bijection

$$G/\mathrm{Stab}_G(x) \longrightarrow G \cdot x$$

(2) For every $\sigma \in G$ and $x \in X$, we have an isomorphism of groups

$$\mathrm{Conj}_\sigma|_{\mathrm{Stab}_G(x)} : \mathrm{Stab}_G(x) \longrightarrow \mathrm{Stab}_G(\sigma \cdot x), \quad g \longmapsto \sigma g \sigma^{-1}$$

(Conjugation by σ)

Proof: (1) Define $f: \alpha(-, x): G \longrightarrow G \cdot x$. Surjective by definition.

$g \longmapsto g \cdot x$

But $g\cdot x = h\cdot x \iff h^{-1}g\cdot x = x \iff h^{-1}g \in \mathrm{Stab}_G(x)$

So this map factors through $G/\mathrm{Stab}_G(x)$ = a set!

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & G\cdot x \\ G/\mathrm{Stab}_G(x) \arrow[ur, "\bar{f}"', "\sim"] & \end{tikzcd}$$

$\bar{f}$ is a bijection

$\bar{f}(g\,\mathrm{Stab}_G) = g\cdot x = f(g)$.

(2) $g \in \mathrm{Stab}_G(x) \iff g\cdot x = x \iff (\sigma g \sigma^{-1})(\sigma x) = \sigma x$
$\iff \sigma g \sigma^{-1} \in \mathrm{Stab}_G(\sigma x)$

Since Conj_σ is an automorphism of G, it induces an isomorphism between $\mathrm{Stab}_G(x)$ & $\mathrm{Stab}_G(\sigma x)$. $\square$

§3 Counting Lemmas

Def: $x \sim_G x'$ in X iff $\exists g \in G$ $g\cdot x = x'$

Claim: This defines an equivalence relation.

$$\boxed{G\backslash X := X/\sim_G \quad \text{equiv classes = orbits}}$$

Easy Observation: $X = \bigsqcup_{\alpha \in G\backslash X} G\cdot x_\alpha \implies |X| = \sum_{\alpha \in G\backslash X} |G\cdot x_\alpha|$

(Here $x_\alpha \in X$ is a choice of an element from the G-orbit labeled by $\alpha \in X/G$)

Recall (Prop) $G/\mathrm{Stab}_G(x) \xrightarrow{\text{bij}} G\cdot x$ & $\mathrm{Stab}_G(x) \xrightarrow{\mathrm{Conj}_\sigma} \mathrm{Stab}_G(\sigma\cdot x)$

Corollary: (a) $|G| = |G\cdot x|\,|\mathrm{Stab}_G x| \quad \forall x \in X$

(b) $|X| = \sum_{\alpha \in G\backslash X} \dfrac{|G|}{|\mathrm{Stab}(x_\alpha)|}$

Next time: Burnside's Lemma for counting orbits.