Lecture 7: Group Actions on Sets II

Recall G a group, X a set, we define a left action of G on X as a map

$$G \times X \longrightarrow X \qquad (g,x) \longmapsto g\cdot x$$

satisfying (i) $e\cdot x = x \quad \forall x \in X$

(ii) $(g_1 g_2)\cdot x = g_1\cdot(g_2\cdot x) \quad \forall g_1,g_2 \in G \ \forall x \in X$

Equivalently: $G \longrightarrow \mathrm{Aut}_{Set}(X)$, $g \longmapsto (x \longmapsto g\cdot x)$ is a group homomorphism. Write $G \circlearrowright X$

Key subsets: orbits, stabilizers & fixed pt sets.

- Orbit of $x\in X$ is $\boxed{G\cdot x} = \{ g\cdot x : g \in G\} \subseteq X$
- Stabilizer of $x \in X$ is $\boxed{\mathrm{Stab}_G x} = \{ g\in G \mid g\cdot x = x\} < G$ (subgroup, but generally ~~not~~ not normal)
- Fix point set for $g\in G$ is $\boxed{X^g} = \{x \in X \mid g\cdot x = x\} \subseteq X$
- Equivalence Relation on X:

$$x \sim_G y \iff \exists g \in G : g\cdot x = y$$
$$\iff x \,\&\, y \text{ in same } G\text{-orbit}$$

$$\rightsquigarrow \boxed{G\backslash X := X/\sim_G} \quad \text{partition of } X \text{ into Equivalence Classes}$$

- Write this partition as $X = \bigsqcup_{x_\alpha \in G\backslash X} G\cdot x_\alpha \implies (*) \ \boxed{|X| = \sum_{x_\alpha \in G\backslash X} |G\cdot x_\alpha|}$

(Here $x_\alpha \in X$ is a choice of an element from the G-orbit labeled by $\alpha \in X/G$)

§1 Left actions of G on itself

① Left Multiplication: $L: G \longrightarrow \mathrm{Aut}_{Set}(G)$, $g \longmapsto L_g$

where $L_g(x) = gx \quad \forall x$

② Right Multiplication $R: G \longrightarrow \mathrm{Aut}_{set}(G)$
$g \longmapsto R_g$

where $R_g(x) = x \cdot g^{-1} \quad \forall x$

Q: Why inverse is necessary? A We want $R_{g_1} \circ R_{g_2} = R_{g_1 g_2}$

$(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$ whereas $g_1 g_2 \neq g_2 g_1$ unless G is abelian.

Similar situation:

$G \circlearrowright X \rightsquigarrow G \circlearrowright \mathrm{Fun}(X, Y)$ via $(g \cdot f)_{(x)} = f(g^{-1} \cdot x)$

Y set $\qquad \qquad \qquad \quad \ni f$

$(g_1 \cdot (g_2 \cdot f))(x) = (g_2 \cdot f)(g_1^{-1} x) = f(g_2^{-1} g_1^{-1} x) = f((g_1 g_2)^{-1} x)$

$= (g_1 g_2 \cdot f)(x)$

$(e \cdot f)(x) = f(x)$ ✓

③ Conjugation (HW1) $C: G \longrightarrow \mathrm{Aut}_{set}(G)$
$g \longmapsto C_g$

where $C_g(x) = g x g^{-1} \quad \forall x.$

§2 Properties of group actions:

There are 3 properties of group actions that are very useful in Diff'l & Alg. Geometry, Rep-n Theory, Topology ("orbifolds", quotient spaces), etc.

(1) Free We say a G-action on X is free if $g \cdot x = x$ for some $x \Rightarrow g = e$

Equivalently
- $\mathrm{Stab}_G(x) = \{e\} \quad \forall x \in X$
- If G is finite, then all orbits have the same size $= |G|$

(2) Transitive: We say a G-action on X is transitive if $\forall x, y \in X$, $\exists g \in G$ such that $g \cdot x = y$

Equivalently: $G \cdot x = X$ for all $x \in X$. (only one orbit)

(3) Faithful: We say a G-action on X is faithful if $G \xrightarrow{\sigma} \mathrm{Aut}_{Set}(X)$ is injective (G is faithfully represented in $\mathrm{Aut}_{Set}(X)$)

Equivalently: $g\cdot x = x \ \ \underline{\underline{\forall}}\, x\in X \implies g=e$

Obs: Free $\implies$ Faithful but Faithful $\not\Rightarrow$ Free

Examples: ① $S_n \circlearrowright \{1,2,\ldots,n\}$
- Faithful ✓ $S_n = \mathrm{Aut}(\{1,\ldots,n\})$
- Free ✗ (S_{n-1} fixes $\{n\}$)
- Transitive ✓ (induct on n)

$\left[S_{n-1} = \mathrm{Stab}_{S_n}(n) \not\trianglelefteq S_n\right]$

② $D_n \circlearrowright \mathbb{R}^2 \setminus \{(0,0)\}$
- Faithful ✓
- Free ✗ ($\exists$ orbits of size $n \neq |D_n|$)
- Transitive ✗ (there are many orbits!)

③ $G \circlearrowright G/H$ by $g(g'H) = gg'H$ Faithful? Free? Transitive? (Exercise)

§3 Counting Lemmas:

Our first objective is to count the orbits of $G \circlearrowright X$

Recall (Prop): $G/\mathrm{Stab}_G(x) \xrightarrow{bij} G\cdot x$ & $\mathrm{Stab}_G(x) \xrightarrow{\text{Conj } \sigma} \mathrm{Stab}_G(\sigma\cdot x)$

Corollary: (a) $|G| = |G\cdot x|\ |\mathrm{Stab}_G x| \quad \forall x\in X$

(b) $|X| = \displaystyle\sum_{x_\alpha \in G\backslash X} \frac{|G|}{|\mathrm{Stab}_G(x_\alpha)|}$

Example: $S_n \circlearrowright \{1,\ldots,n\}$ Only 1 orbit $= X$.

$\mathrm{Stab}_{S_n}(n) \simeq S_{n-1} \hookrightarrow S_n \implies n = |X| = \dfrac{|G|}{|\mathrm{Stab}_G(n)|} = \dfrac{n!}{(n-1)!}$ □

Our first main result is the following:

Burnside Lemma (Frobenius, 1887) # of orbits = average # of fixed pts

More precisely: $\left|G\backslash X\right| = \frac{1}{|G|}\sum_{g\in G}|X^g| = \frac{1}{|G|}\sum_{x\in X}|\mathrm{Stab}_G(x)|$

Proof: Write $F := \{(g,x)\in G\times X \mid g\cdot x = x\} \subseteq G\times X$

$$\begin{tikzcd} & F \arrow[dl, "\pi_1"'] \arrow[dr, two heads, "\pi_2"] & \\ G & & X \end{tikzcd}$$

(incidence corresp.)

Claim 1: $|F| = \sum_{g\in G}|X^g|$

Pf/ Sum over 1st component (Fix g: $(g,x)\in F \Leftrightarrow x\in X^g$)

Claim 2: $|F| = \sum_{x\in X}|\mathrm{Stab}_G(x)|$

Pf/ Sum over 2nd component (Fix x: $(g,x)\in F \Leftrightarrow g\in\mathrm{Stab}_G(x)$)

Combining the two claims: $X = \bigsqcup_{x_\alpha\in G\backslash X} G\cdot x_\alpha$

$$|F| = \underbrace{\sum_{g\in G}|X^g|} = \sum_{x\in X}|\mathrm{Stab}_G(x)| \overset{\downarrow}{=} \sum_{x_\alpha\in G\backslash X}\ \sum_{y\in G\cdot x_\alpha}\underbrace{|\mathrm{Stab}_G(y)|}_{=|\mathrm{Stab}_G(x_\alpha)|}$$

$$= \sum_{x_\alpha\in G\backslash X}\underbrace{|\mathrm{Stab}_G(x_\alpha)||G\cdot x_\alpha|}_{=|G|} = \underbrace{|G|\,|G\backslash X|} \qquad (y = \sigma\cdot x_\alpha)$$

So $\frac{1}{|G|}\sum_{g\in G}|X^g| = |G\backslash X|$ ✓ □

§4 Applications

① Pick n & an composition of $n = (a_1,\ldots,a_r)$ $a_1+\cdots+a_r = n$ $a_i\in\mathbb{Z}_{>0}$

$\Rightarrow X$ = set of all ordered partitions $P_1\sqcup\cdots\sqcup P_r$ of $\{1,\ldots,n\}$ with $|P_i| = a_i$.

Eg $n = 7 = 3+2+2$ $|P_1| = 3$, $|P_2| = 2$, $|P_3| = 2$

Then $S_n \circlearrowright X$ & this action is transitive (one orbit!)

$\mathrm{Stab}_{S_n}(\underline{x}) \cong S_{a_1}\times S_{a_2}\times\cdots\times S_{a_r}$ (with coordwise multiplication)

$$|S_n \backslash X| = 1 = \frac{1}{|S_n|} \sum_{x\in X} |\mathrm{Stab}_{S_n}(x)| = \frac{1}{n!} \sum_{x\in X} a_1! a_2! \cdots a_r!$$
$$= \frac{1}{n!} |X| \, a_1! \cdots a_r!$$

Eg: $\mathrm{Stab}_{S_n}(\{1,2,3\} \sqcup \{4,5\} \sqcup \{6,7\}) \simeq S_3 \times S_2 \times S_2$

$$\Rightarrow |X| = \frac{|S_7|}{|\mathrm{Stab}_{S_7}(x)|} = \frac{7!}{3!\,2!\,2!}$$

In general, we get the formula for the multinomial coeff!

$$\boxed{\# \text{ partitions of } \{1,\dots,n\} \text{ with } r \text{ parts of sizes } (\lambda_1,\dots,\lambda_r) \; (|P_i| = \lambda_i \;\forall i \quad \lambda_1+\dots+\lambda_r = n) = \frac{n!}{\lambda_1!\cdots\lambda_r!}}$$

(2) Consider $G \circlearrowright G$ by conjugation

For $x\in G$, the stabilizer $= \{g\in G \mid gxg^{-1} = x\}$ is also called the centralizer of x. We denote it by $Z_G(x)$.

- G-orbits under conjugation are called conjugacy classes. Write the set of all these classes by $\mathcal{C}$.

Obs: For each $g\in G$, the set of elements of G fixed under C_g is $Z_G(g)$

(Indeed: $ghg^{-1} = h \Leftrightarrow g^{-1}hg = h \Leftrightarrow hg = gh \Leftrightarrow hgh^{-1} = g \Leftrightarrow h\in Z_G(g)$)

By our counting lemmas:

- $|G| = \sum_{x_\alpha\in\mathcal{C}} |G\cdot x_\alpha| = \sum_{x_\alpha\in\mathcal{C}} \left|\frac{G}{Z_G(x_\alpha)}\right| = \sum_{x_\alpha\in\mathcal{C}} [G : Z_G(x_\alpha)]$.
- $\underbrace{|\mathcal{C}|}_{\#\text{ conj. classes}} = \frac{1}{|G|}\sum_{g\in G} |G^g| \underset{\text{by Obs}}{=} \underbrace{\frac{1}{|G|}\sum_{g\in G} |Z_G(g)|}_{\text{average }\#\text{ of elements in a centralizer}}$