Lecture 10: Semidirect products

Last Time: Used Sylow Theorems to study non-simple groups of a fixed order
- Classified groups of order p^2 with $p>0$ prime

TODAY: New tool for classifying groups.

§1 Motivation:

- **HW1**: Saw all groups of order ≤ 5 are abelian; S_3, D_3 are not.
- **Goal**: Classify groups G of order $6 = 2\cdot 3$

By Sylow Thm: $\left.\begin{array}{l} n_2 \equiv 1 \mod (2) \\ n_2 \mid 3 \end{array}\right\} n_2 = 1 \text{ or } 3$; $\left.\begin{array}{l} n_3 \equiv 1 \mod 3 \\ n_3 \mid 2 \end{array}\right\} \Rightarrow \boxed{n_3 = 1}$

$Syl_3(G) = \{Q\}$ with $Q = \langle h \rangle \simeq \mathbb{Z}/3\mathbb{Z}$; $Syl_2(G) \ni P = \langle g \rangle \simeq \mathbb{Z}/2\mathbb{Z}$

CASE 1: $n_2 = 1$ Then, $Syl_2(G) = \{P\}$

Since $P, Q \lhd G$ & $P \cap Q = \{e\}$ $\underset{HW1}{\Longrightarrow}$ P & Q mutually commute.

$$\begin{array}{ccc} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \simeq P \times Q & \longrightarrow & G \\ (k, \ell) \mapsto (g^k, h^\ell) & \longrightarrow & g^k h^\ell \end{array}$$

is an injective gp homomorphism

$|G| = |P \times Q| = 6$, so iso

$$G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

CASE 2: $n_2 = 3$ Then, $Syl_2(G) = \{\overset{\overset{P}{\shortparallel}}{P_1}, P_2, P_3\}$

Obs: h & g cannot commute (Otherwise $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ so $n_2 = 1$)

$G = \{1, h, h^2, g, gh, gh^2\}$ is a group so $hg = gh^2$ so $ghg^{-1} = ghg = h^2$

Conclude: $G \simeq D_3$ where $\begin{array}{l} h \longmapsto \rho \\ g \longmapsto s \end{array}$ (same relations!) & $D_3 \simeq S_3$ $\begin{array}{l} \rho \mapsto (123) \\ s \mapsto (12) \end{array}$

$[(12)(123)(12) = (132)]$

- We can view the construction more generally!

(1) We let P act on $Q \lhd G$ by conjugation:

$$\begin{array}{cccl} \alpha : P & \longrightarrow & Aut_{Set}(Q) & \text{gp hom} \\ g^i & \longmapsto & (g^i h^j g^{-i} = \begin{cases} h^{2j} & \text{if } i=1 \\ h^j & \text{if } i=0 \end{cases}) \end{array}$$

$i=0$: $h^j \xrightarrow{\alpha(e)} h^j$

$i=1$: $h^{j+k} \xrightarrow{\alpha(g)} \underset{\alpha_{(g)}(h^j)\, \alpha_{(g)}(h^k)}{\overset{\shortparallel}{h^{2(j+k)}}}$

So $\alpha : P \longrightarrow Aut_{Gp}(Q)$ is group hom.

② $G = PQ = \{g^i h^j\} = QP$ & $Q \lhd G$ with $P \cap Q = \{e\}$. (saw in 3rd Iso Thm)

- The map α provides the "commutation relation" between $g \in P$ & $h \in Q$.

§2. Semidirect Products I

We will give 3 equivalent constructions, starting from ②.

Definition: We say a group G is a semi-direct product of two subgroups H & N if

(i) $H \leq G$ & $N \trianglelefteq G$

(ii) $G = HN = \{h.n \;\; h \in H, n \in N\} = NH$ [Write: $G = N \rtimes H$] \rtimes

(iii) $H \cap N = \{e\}$

Obs: Recall 3rd Iso Theorem:

$H < G$, $N \lhd G$ $\rightsquigarrow$

$$\begin{tikzcd}[column sep=small]
 & G \arrow[d, dash, "\vee"] & \\
 & HN = NH \arrow[dl, dash, "<"'] \arrow[dr, dash, "\rhd"] & \\
H \arrow[dr, dash, "\lhd"'] & & N \arrow[dl, dash, "<"] \\
 & H \cap N &
\end{tikzcd}$$

(= Today)

& $\underset{\text{(today)}}{\dfrac{H}{e}} = \dfrac{H}{H \cap N} \xrightarrow{\sim} \dfrac{HN}{N} = \underset{\text{(today)}}{\dfrac{G}{N}}$

Consequence: $H \simeq G/N$, so for each coset $gN \in G/N$ we can find a representative $\sigma_g \in H$ so that $\sigma_{g_1 g_2} = \sigma_{g_1} \cdot \sigma_{g_2}$.

Example 1: $H \lhd G$, then $G \simeq N \times H$ (direct product) (coord. wise structure)

Example 2 $G = D_n = \langle s, \rho \rangle$; $H = \{e, s\} < G$, $H \cong \mathbb{Z}/2\mathbb{Z}$; $N = \langle \rho \rangle \lhd G$, $N \cong \mathbb{Z}/n\mathbb{Z}$ $(s\rho^j s^{-1} = \rho^{-j})$

$\Rightarrow D_n \simeq \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$

Example 3: $G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in GL_2(\mathbb{C}) \right\}$; $\mathbb{C} \simeq N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right\} \lhd G$

$(\mathbb{C}^*)^2 \simeq H = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \in G \right\} < G$

- $H \cap N = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
- $G = HN = NH$ because $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \underset{\in H}{\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}} \underset{\in N}{\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}} = \underset{\in N}{\begin{bmatrix} 1 & b/c \\ 0 & 1 \end{bmatrix}} \underset{\in H}{\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}}$.

$\Rightarrow G \simeq \mathbb{C} \rtimes (\mathbb{C}^*)^2$

§3. Semidirect Products II:

Fix two groups H & N and a group homomorphism

$$\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N) = \{\text{isos}: N \xrightarrow{\sim} N\}$$

Obs: $\alpha(h_1 h_2) = \alpha_{(h_1)} \circ \alpha_{(h_2)}$, $\alpha(e) = id_N$, $\alpha(h^{-1}) = \alpha(h)^{-1}$

We can use this to define a binary operation of the cartesian product $N \times H$.

$$G = \{(n, h) : n \in N, h \in H\}$$

$$\boxed{(n_1, h_1) * (n_2, h_2) = (n_1 \cdot \alpha_{(h_1)}(n_2), h_1 h_2) \quad \forall\, n_1, n_2 \in N,\ h_1, h_2 \in H.}$$

Lemma: This operation defines a group structure on G. Write $\boxed{G = N \rtimes_\alpha H}$ (H acts on N via α)

Pf/ We need to check associativity, neutral element & inverses:
Next Time!