Lecture 11: Semidirect Products II & Short Exact Sequences

Last Time: 2 characterizations of semidirect products

1. $H < G$, $N \lhd G$

   $G = NH \ (= HN)$

   $H \cap N = \{e\}$

   $\rightsquigarrow G = N \rtimes H$

   (Eg: $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$)

2. $\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N)$ gp hom.

   $G = N \times H$ as a set with group operation

   $(n_1, h_1) \cdot (n_2, h_2) = (n_1\, \alpha(h_1)(n_2),\ h_1 h_2)$

   $\rightsquigarrow G = N \rtimes_\alpha H$

   (Eg: $N \lhd G$, $H \leq H$, $\alpha : H \to \mathrm{Aut}_{Gp}(N)$ by conjugation)

Today: • How to relate them?

1 $\Longrightarrow$ 2

$$\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N)$$
$$h \longmapsto (g \mapsto h g h^{-1})$$

2 $\Longrightarrow$ 1

$$N \hookrightarrow G, \quad n \longmapsto (n, e_H) \qquad , \qquad H \hookrightarrow G, \quad h \longmapsto (e_N, h)$$

group homomorphisms injective.

• One more characterization: using split short exact sequences

§1 Semidirect Products II:

Fix two groups H & N and a group homomorphism

$$\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N) = \{\text{isos}: N \xrightarrow{\sim} N\}$$

Obs: $\alpha(h_1 h_2) = \alpha(h_1) \circ \alpha(h_2)$, $\alpha(e) = \mathrm{id}_N$, $\alpha(h^{-1}) = \alpha(h)^{-1}$

We can use this to define a binary operation of the cartesian product $N \times H$.

$$G = \{(n, h) : n \in N,\ h \in H\}$$

$$\boxed{(n_1, h_1) * (n_2, h_2) = (n_1 \cdot \alpha(h_1)(n_2),\ h_1 h_2) \qquad \forall\, n_1, n_2 \in N,\ h_1, h_2 \in H.}$$

Lemma: This operation defines a group structure on G. Write $\boxed{G = N \rtimes_\alpha H}$

(H acts on N via α)

Pf/ We need to check associativity, neutral element & inverses:

① **Associativity:**

$g_1=(n_1,h_1),\quad g_2=(n_2,h_2),\quad g_3=(n_3,h_3)\in G$

$g_2\cdot g_3 = (n_2\cdot\alpha(h_2)(n_3),\ h_2h_3)\quad ;\quad g_1g_2=(n_1\alpha(h_1)(n_2),\ h_1h_2)$

$$g_1(g_2g_3) = (n_1\ \alpha(h_1)(n_2\alpha(h_2)(n_3)),\ h_1(h_2h_3))$$

α gp hom. / $\alpha(h_1)$ gp hom.

$$= (n_1\ \alpha(h_1)(n_2)\cdot\alpha(h_1h_2)(n_3),\ (h_1h_2)h_3)$$
$$= ((n_1,h_1)(n_2,h_2))\cdot(h_3,n_3) = (g_1g_2)g_3$$

② **Neutral Element:** (e_N,e_H)

Why? $(n_1,h_1)\cdot(n_2,h_2) = (n_1,h_1)$

$(n_1\alpha(h_1)(n_2),\ h_1h_2) = (n_1,h_1)\ \forall h_1$ forces $h_2=e_H$ & $\alpha(h_1)(n_2)=e_N\ \forall h_1$

In particular for $h_1=e_H$, so $n_2=e_N$.

Check: $(n_1,h_1)(e_N,e_H) = (n_1\underbrace{\alpha(h_1)(e_N)}_{=e_N},\ h_1e_H) = (n_1,h_1)$ ✓

$(e_N,e_H)(n_1,h_1) = (e_N\underbrace{\alpha(e_H)}_{id}(n_1),\ e_Hh_1) = (n_1,h_1)$ ✓

③ **Inverses:** $(n,h)^{-1} = (\alpha(h^{-1})(n^{-1}),\ h^{-1})$

Why? $(n_1,h_1)(n_2,h_2) = (n_1\alpha(h_1)(n_2),\ h_1h_2) = (e_N,e_H)$ forces $h_2=h_1^{-1}$

& $\alpha(h_1)(n_2) = n_1^{-1}$ so $n_2 = \alpha(h_1)^{-1}(n_1^{-1}) = \alpha(h_1^{-1})(n_1^{-1})$

Check: $(n,h)(\alpha(h^{-1})(n^{-1}),\ h^{-1}) = (n\underbrace{\alpha(h)(\alpha(h^{-1})}_{id}(n^{-1})),\ hh^{-1}) = (e_N,e_H)$ ✓

$$(\alpha(h^{-1})(n^{-1}),h^{-1})(n,h) = (\alpha(h^{-1})(n^{-1})\ \alpha(h^{-1})(n),\ h^{-1}h)$$
$$= (\alpha(h^{-1})(\underbrace{n^{-1}n}_{=e_N}),\ e_H) = (e_N,e_H)$$ ✓

$\alpha(h^{-1})$ gp hom!

□

Next, we relate $N\rtimes_\alpha H$ to the construction from 1st characterization:

Proposition: The maps $\begin{array}{ccc} H & \longrightarrow & G=N\rtimes_\alpha H \\ h & \longmapsto & (e_N,h)\end{array}$; $\begin{array}{ccc} N & \longrightarrow & G=N\rtimes_\alpha H \\ n & \longmapsto & (n,e_H)\end{array}$

are injective group homomorphisms. Furthermore:

(i) $H\leq G$, $N\trianglelefteq G$ (via the injections) (ii) $NH=G$

(iii) $H\cap N = \{(e_N,e_H)=:e_G\}$ So $G\cong N\rtimes H$

Proof: Check the structures of H & N are compatible with that of G

H: $(e_N,h_1)(e_N,h_2) = (e_N \underbrace{\alpha(h_1)(e_N)}_{=e_N}, h_1h_2) = (e_N, h_1h_2)$

$$\Rightarrow \quad \begin{aligned} H &\xrightarrow{\varphi_H} G \\ h &\longmapsto (e_N,h) \end{aligned}$$ is group homomorphism. Clear: $\mathrm{Ker}(\varphi_H)=\{e_H\}$

N $(n_1,e_H)(n_2,e_H) = (n_1 \underbrace{\alpha(e_H)}_{=id_N}(n_2), e_He_H) = (n_1n_2, e_H)$

$$\Rightarrow \quad \begin{aligned} N &\xrightarrow{\varphi_N} G \\ n &\longmapsto (n,e_H) \end{aligned}$$ is gp hom. Clear: $\mathrm{Ker}(\varphi_N)=\{e_N\}$.

(i) $H \text{ "="} \mathrm{Im}(\varphi_N) \leq G$

Claim: $N \text{ "="} \mathrm{Im}(\varphi_N) \trianglelefteq G$

Pf/ $(n,h)\, N\, (n,h)^{-1} = (n,h)\, N\, (\alpha(h^{-1})(n^{-1}), h^{-1}) = N$

$(n,h)(n_1,e_H)(\alpha(h^{-1})(n^{-1}), h^{-1}) = (n\,\alpha(h)(n_1), h)(\alpha(h^{-1})(n^{-1}), h^{-1})$

$= (n\,\alpha(h)(n_1)\,\underbrace{\alpha(h)(\alpha(h^{-1})(n^{-1}))}_{=n^{-1}}, hh^{-1}) = (n\,\alpha(h)(n_1)\,n^{-1}, e_H) \in N$.

(ii) $NH = G$ by construction

$(n,e_H)(e_N,h) = (n\underbrace{\alpha(e_H)(e_N)}_{=e_N}, e_H h) = (n,h)$

(iii) $N \cap H = \{(n,h) : (n,e_H) = (e_N,h)\} = \{(e_N,e_H) = e_G\}$. □

Conversely the construction $N \rtimes H$ always arises from some $N \rtimes_\alpha H$.

Prop: Given $G = N \rtimes H$, we set $\alpha: H \longrightarrow \mathrm{Aut}_{Gp}(N)$

$$h \longmapsto (n \longmapsto hnh^{-1})$$

Then, $\Phi: N \rtimes_\alpha H \longrightarrow G$ is an isomorphism of groups

$$(n,h) \longmapsto nh$$

Proof: It is easy to check that α is gp homomorphism, so $N \rtimes_\alpha H$ is well-defined.

Claim 1: Φ is a gp homomorphism.

$$\Phi((n_1,h_1)(n_2,h_2)) = \Phi((n_1\alpha(h_1)(n_2),\ h_1h_2))$$
$$= \underbrace{n_1\alpha(h_1)(n_2)}_{\in N}\ \underbrace{h_1h_2}_{\in H}$$
$$\overset{\text{def of }\alpha}{=} n_1(h_1n_2h_1^{-1})h_1h_2 = n_1h_1n_2h_2$$
$$= \Phi(n_1,h_1)\,\Phi(n_2,h_2).$$

Claim 2 $\operatorname{Ker}\Phi = \{(e_N,e_H)\}$ (because $H\cap N=\{e\}$)

Claim 3: $\operatorname{Im}\Phi = G$ (because $NH=G$)

By the 3 Claims, Φ is gp isomorphism. □

Obs: Even though conjugation gives $N\rtimes H$, we might be able to use a different $\alpha' : H \to \operatorname{Aut}_{Gp} N$. since different gp hom can give rise to isomorphic gps.

Summary: (not 1-to-1 !)

Semidirect Products of N & H $\longleftrightarrow$ Gp hom $\alpha: H \to \operatorname{Aut}_{Gp}(N)$

$N\rtimes_\alpha H \longleftarrow\!\!| \ \alpha$

Obs: These constructions generalize to Hopf algebras. ("Quantum Double Constructions")

Missing: 3rd characterization via split short exact sequences.

§2. Short Exact Sequences:

Recall (1st Isomorphism Theorem) $\varphi\ G \twoheadrightarrow G'$ gp hom, then $G/\operatorname{Ker}\varphi \overset{\bar{\varphi}}{\simeq} G'$.

This statement is often written as:

Theorem: We have an exact sequence (see definition below):

$$1 \longrightarrow \operatorname{Ker}(\varphi) \xrightarrow{i} G \xrightarrow{\varphi} G' \longrightarrow 1$$

where:
- $1 = \{1\}$ is the trivial group
- $i : \operatorname{Ker}(\varphi) \longrightarrow G$ is the natural inclusion
- $1 \longrightarrow \operatorname{Ker}(\varphi) : 1 \mapsto e_G$ & $G' \longrightarrow 1 : g' \mapsto 1\ \forall g' \in G'$.

Definition: A sequence of group homomorphisms

$$G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\Psi} G_3$$

is said to be exact (or exact at G_2) if $\operatorname{Im}\varphi = \operatorname{Ker}\Psi$.

Obs: $\operatorname{Ker}\Psi \triangleleft G_2$ but in general $\operatorname{Im}\varphi$ is not (unless G_2 is abelian)
So this is a strong condition to impose!

Examples: ① $1 \longrightarrow G_1 \xrightarrow{\Psi} G_2$ is exact $\iff$ Ψ is injective
② $G_1 \xrightarrow{\varphi} G_2 \longrightarrow 1$ is exact $\iff$ φ is surjective.

Def: An exact sequence of the form

$$1 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\Psi} G_3 \longrightarrow 1$$

is usually referred to as a short exact sequence (ses). It signifies that:

(i) G_1 can be viewed as a normal subgroup of G_2 because $G_1 \xrightarrow{\sim} \underset{\operatorname{Ker}\Psi \triangleleft G_2}{\overset{\parallel}{\operatorname{Im}\varphi}} < G_2$

(ii) $G_2/\operatorname{Im}\varphi = G_2/\operatorname{Ker}\Psi \xrightarrow[\sim]{\overline{\Psi}} G_3$ is an iso.

Ex: 1st Isomorphism Theorem:

$G \xrightarrow{\varphi} G'$ gp hom $\rightsquigarrow$

$$\begin{tikzcd}
1 \arrow[r] & \operatorname{Ker}\varphi \arrow[r] & G \arrow[r, "\varphi"] & \operatorname{Im}(\varphi) \arrow[r] & 1 \\
1 \arrow[r] & \operatorname{Ker}\varphi \arrow[r] \arrow[u, equal, "\cong\ \mathrm{id}"] & G \arrow[r, "\pi"] \arrow[u, equal, "\cong\ \mathrm{id}"] & G/\operatorname{Ker}\varphi \arrow[r] \arrow[u, "\cong", "\overline{\varphi}"'] & 1
\end{tikzcd}$$

Obs: These 2 short exact sequences are called equivalent (vertical maps should be isos.

The exact sequence $1 \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\Psi} G_3 \longrightarrow 1$ also signifies that we can "build G_2 out of G_1 & G_3". More precisely "G_2 is an extension of G_3 by G_1"

§3. Sections & Retractions:

Fix
$$\begin{tikzcd}
1 \arrow[r] & G_1 \arrow[r, "\varphi"] & G_2 \arrow[l, dashed, bend left, "\exists r?"] \arrow[r, "\Psi"] & G_3 \arrow[l, dashed, bend left, "\exists s?"] \arrow[r] & 0
\end{tikzcd}$$
ses

Q: Can we use $G_1 \triangleleft G_2$ & G_3 to understand/characterize G_2?

A: Usually knowing $N \triangleleft G$ & G/N does not characterize G !
(see HW4).

Definition: A ses is split if we have a section, that is, a gp hom
$s: G_3 \longrightarrow G_2$ with $\Psi \circ s = id_{G_3}$ (=> s is injective!)

Definition: A ses is trivial if we have a retraction (or projection), that is, a gp hom
$r: G_2 \longrightarrow G_1$ with $r \circ \varphi = id_{G_1}$. (=> r is surjective!)

Lemma: A trivial ses always splits

Proof: next time.

Upshot:
- trivial ses ⇝ G_2 is a direct product of G_1 & G_3
- split ses ⇝ G_2 is a semidirect product of G_1 & G_3

$G_2 \simeq G_1 \rtimes_\alpha G_3$ $G_3 \leq G_2$ via s

Use s to build $\alpha: G_3 \to \mathrm{Aut}_{Gp}(G_1)$ gp hom.