Lecture 12: Short exact sequences, Composition Series

§1 Short exact sequences:
Recall a short exact sequence (ses) is a sequence of the form
\[ 1 \rightarrow G_1 \xrightarrow{\psi} G_2 \xrightarrow{\Psi} G_3 \rightarrow 1 \]
involving groups and group homomorphisms, with
(i) \( \psi \) injective
(ii) \( \ker \Psi = \text{Im} \psi \)
(iii) \( \Psi \) surjective.

Definition: A ses is split if we have a section, that is, a gp hom
\( s: G_3 \rightarrow G_2 \) with \( \Psi \circ s = \text{id}_{G_3} \)
(\( \Rightarrow \) s is injective!)

Definition: A ses is trivial if we have a retraction, that is, a gp hom
\( r: G_2 \rightarrow G_1 \) with \( r \circ \Psi = \text{id}_{G_1} \).
(\( \Rightarrow r \) is surjective!)

See HW#4 for examples. In particular,

Obs 1: Not every ses splits!
Ex: \[ 1 \rightarrow \{\pm 1\} \rightarrow \mathbb{Q}_3 \rightarrow \mathbb{Q}_8 \rightarrow \{\pm 1\} \rightarrow 1 \]
why elements of order 2 are \( \pm 1 \) all elements have order 2 (except \( \pm 1 \))

Obs 2: trivial and split ses are different things!
Ex: \[ 1 \rightarrow \mathbb{Z} \rightarrow \{\pm 1\} \rightarrow \mathbb{Q}_8 \rightarrow \{\pm 1\} \rightarrow 1 \]

Claim 1: \( S(-1) = (12) \) satisfies \( s \circ \psi \circ s = \text{id}_{\{\pm 1\}} \) \( \Rightarrow \) ses splits

Claim 2: \( \Psi r : S_3 \rightarrow \mathbb{A}_3 \) gp hom s.t. \( r \circ i = \text{id}_{\mathbb{A}_3} \)
Why? set \( \sigma = [(ij) (ifj)] \) \( o((ij)) = 2 \) but \( o(\sigma) \mid |A_3| = |S_3| = 3 \)
so \( o(\sigma) = 1 \).

But every permutation in \( S_3 \) is a product of transpositions so \( \sigma \) must be trivial in \( S_3 \): \( \text{Im} r = 1 \leq A_3 \) Contr! since \( r \) is surjective \( \square \)
Lemma: A trivial ses always splits

\[ \begin{array}{cccccc}
0 & \rightarrow & A & \stackrel{\alpha}{\rightarrow} & B & \stackrel{\beta}{\rightarrow} & C & \rightarrow & 1
\end{array} \]

\[ A, B, C \text{ g.p.s.} \]

\[ \gamma : B \rightarrow A \quad \text{co} \gamma = 1d_A \]

Want to build a gp hom \( s : C \rightarrow B \) with \( s \circ \gamma = 1d_C \).

We write \( \text{Ker } r \rightarrow \text{Im } \gamma \rightarrow C \) gp homomorphism.

Claim 1: \( \Psi|_{\text{Ker } r} \) is injective.

3f/ Pick \( b \in \text{Ker } r \) with \( \Psi(b) = e_C \). so \( b \in \text{Ker } \Psi = \text{Im } \gamma \)

so \( b = \psi(a) \) for \( a \in A \).

Then \( e_A = c(b) = co\psi(a) = a \)

\[ \Rightarrow b = \psi(e_A) = e_B \]

Claim 2: \( \Psi^{-1} \) is injective

3f/ Given \( c \in C \) pick \( b \in B \) with \( \Psi(b) = c \). This choice is not unique, but if \( \Psi(b') = c \) then \( b' = b\psi(a) \) for \( a \in A \).

Pick \( b' = b\psi(\sigma(b')) \). Note: \( b' \in \text{Ker } r \) because

\[ r(b') = r(b) \cdot co\psi(\sigma(b')) = r(b) \cdot r(\sigma(b')) = e_B \]

Also \( \Psi(b') = \Psi(b) \)

Then \( \exists s : C \rightarrow \text{Ker } r \rightarrow B \) gp homomorphism

with \( s \circ \gamma = 1_C \). \( \Rightarrow \) the ses splits. \( \square \)

§2. Direct / Semidirect Products and s.e.s.

Split & Trivial ses will characterize \( G_2 \) as \( G_1 \times G_2 \) or \( G_1 \rtimes G_2 \)

Proposition 1: If the ses \( \begin{array}{cccccc}
0 & \rightarrow & N & \stackrel{\phi}{\rightarrow} & G & \stackrel{\psi}{\rightarrow} & H & \rightarrow & 0
\end{array} \)

is trivial, then \( G \cong N \times H \) (direct product) when \( N \hookrightarrow G \) & \( H \hookrightarrow G \)
Proof: Assume \( \exists \; \varphi : G \to N \) retraction. Then, by Lemma (page 2),
we have a section \( H \hookrightarrow G \).

The s.e.s. for the triple \((N, N \times H, H)\) is trivial.

We have \( 0 \to N \xrightarrow{i} N \times H \xrightarrow{\pi_2} H \to 0 \)
\[ \xymatrix{ 0 \\ H \\ N \times H \\ N \\ H \ar[u]_{\pi_2} \ar[l]_{\pi_1} \\ N } \]

Define \( \eta : G \to N \times H \) via \( \eta(g) = (\varphi(g), \Psi(g)) \).

- \( \eta \) is surjective since both \( \varphi \) and \( \Psi \) are.

- **Claim:** \( \eta \) is surjective.

*Proof:* Pick \( x \in N \) and \( h \in H \). Choose \( g \in G \) with \( \Psi(g) = h \) (exists because \( \Psi \) is surjective).

Take \( \breve{g} = g \cdot (\varphi(g)^{-1}) \cdot \Psi(x) \in G \),

\[ \Rightarrow \Psi(\breve{g}) = \Psi(g) \cdot (\varphi(g)^{-1}) \cdot \Psi(x) = \Psi(g) \cdot \Psi(g^{-1}) \cdot \Psi(x) = e_H \cdot e_N \] \[ = e_H \cdot e_N = e_G \] \[ \Rightarrow \eta(\breve{g}) = (x, h) \]

**Claim 2:** \( \eta \) is injective.

*Proof:* If \( \eta(g) = (e_N, e_H) \) then \( \Psi(g) = e_H \), so \( g \in \ker \Psi = \operatorname{Im} \varphi \).

Then, \( \exists \; x \in N \) with \( g = \varphi(x) \)

\[ \Rightarrow e_N = \varphi(g) = \varphi(\varphi(x)) = x \]

It is easy to check all squares commute.

Proposition 2: If a s.e.s. \( 0 \to N \xrightarrow{\varphi} G \xrightarrow{\Psi} H \to 0 \) splits, then
\( G \cong N \times H \) where \( N \cong G \) and \( H \hookrightarrow G \).
Theorem: \( N \triangleleft G \) \& \( H \leq G \).

Claim 1: \( s(H) \cap \Phi(N) = \{ e \} \)

Pick \( g \in s(H) \cap \Phi(N) \) then \( g = s(h) = \Psi(x) \) \( x \in N, h \in H \)

\[ \Rightarrow \Psi(g) = \Psi \circ s(h) = h \]
\[ \Rightarrow g = s(e_H) = e_G \]
\[ = \Psi \circ \Psi(x) = e_H \]

Claim 2: \( N H = \{ \Psi(x) s(h) \mid x \in N, h \in H \} = G \)

Pick \( g \in G \) \( \Rightarrow \Psi(g) \in H \)

Pick \( \hat{g} = (s \circ \Psi)(g) \). It satisfies \( \Psi(g) = \Psi(\hat{g}) \), so

\[ \hat{g}^{-1} g \in ker \Psi = \text{Im } \Psi \] so \( \hat{g}^{-1} g = \Psi(x) \) for some \( x \in N \).

Then \( g = \hat{g} \Psi(x) = s \circ \Psi(g) \Psi(x) = s \circ \Psi(g) \Psi(x)(s \circ \Psi(g))^{-1}(s \circ \Psi(g)) \)

\[ \triangledown \]

By definition, \( G = \Psi(N) \times \Psi(H) \triangleleft N \times H \).

Example: \( \mathbb{A} \xrightarrow{1} \mathbb{A}_n \xrightarrow{\iota} \mathbb{S}_n \xrightarrow{s \circ \Psi} \mathbb{Z} \left\{ \pm 1 \right\} \xrightarrow{1-1} \mathbb{A} \) splits

(\text{HW 4})

so \( \mathbb{S}_n = \mathbb{A}_n \times \mathbb{Z}/2\mathbb{Z} \).

§3 Composition Series

Recall: A group \( S \) is called simple if \( \{ e \} \leq S \) are the only normal subgroups of \( S \).

Examples: \( \mathbb{A}_n \) for simple \( (\text{next week}) \)

\( \mathbb{Z}/p\mathbb{Z} \quad p > 0 \) prime are simple

\( \text{PSL}_n = \mathbb{S}_n / \mathbb{Z}(\mathbb{S}_n) \) are simple
Def. A composition series of a group $G$ is a finite sequence of subgroups of $G$

$$\Sigma : \quad G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_k = \{e\}$$

such that $G_{j+1} \triangleleft G_j$ is normal for all $j = 0, \ldots, k-1$.

The successive quotients: $q_{j+1}^j(G) = G_j / G_{j+1}$

(Other notation: $q_{j+1}^j(G)$ if $\Sigma$ is not clear from context.)

Def. A composition series $\Sigma'$ is said to refine $\Sigma$ (or be finer than $\Sigma$) if $\Sigma$ is obtained from $\Sigma'$ by omitting some terms.

More precisely: $\Sigma' : \quad G = G'_0 \supseteq G'_1 \supseteq \ldots \supseteq G'_m = \{e\}$

$\Sigma : \quad G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_n = \{e\}$

$\Sigma'$ is finer than $\Sigma$ if $n \leq m$ and there exists an order-preserving injective map $\Phi : \{0, \ldots, m\} \to \{0, \ldots, n\}$ with $G_j = G'_{\Phi(j)}$.

Ex. $\Sigma : \quad G = \mathbb{Z}/6\mathbb{Z} \supseteq \mathbb{Z}/3\mathbb{Z} \supseteq \{e\}$ is a refinement, only coarsening

$\Sigma' : \quad G = \mathbb{Z}/6\mathbb{Z} \supseteq \mathbb{Z}/2\mathbb{Z} \supseteq \{e\}$

$h_0^{\Sigma'}(G) = \mathbb{Z}/6\mathbb{Z} / \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} = \eta_1^{\Sigma'}(G)$, $h_1^{\Sigma'} = \mathbb{Z}/2\mathbb{Z} = \eta_0^{\Sigma'}(G)$.

$\Sigma_0 : \quad \mathbb{Z}/6\mathbb{Z} \supseteq \{e\}$, $\Sigma_1$ refines $\Sigma$ via $\Phi : \{0,1\} \to \{0,1\}$ $\phi(0) = 1 \phi(1) = 2$

Remark: In general, a series obtained from a composition series $\Sigma'$ by omitting some terms is not a composition series since for $j > i + 1$,

$G'_j$ is not in general a normal subgroup of $G'_{j+1}$.

Ex. $G = D_4 \supseteq \langle p^2, s \rangle \supseteq \langle s \rangle \supseteq \{e\}$

$G_1 = \langle p^2, s \rangle \triangleleft G$ \quad $p^2p^{-1} = p^2 \in G_1$ \quad $e \neq p^2 \in G_1$

$G_2 = \langle s \rangle \triangleleft \langle p^2, s \rangle$ \quad $p^2sp^{-2} = p^2ps = p^4s = s \in G_1$

$G_3 = \{e\} \triangleleft \langle s \rangle$ \quad $ss^{-1} = s \in G_1$
\[n_0(D_4) = \langle p, s \rangle \cong \langle p \rangle \cong \mathbb{Z}/2\mathbb{Z}\]
\[n_1(D_4) = \langle p^2, s \rangle \cong \langle p^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}\]
\[n_2(D_4) = \langle s \rangle \cong \mathbb{Z}/2\mathbb{Z}\]

We can't omit \(\langle p^2, s \rangle\) and have a composition series because \(\langle s \rangle \neq D_4\).

We can omit \(\langle s \rangle\) and get a comp series
\[\Sigma_2: D_4 \geq \langle p^2, s \rangle \geq \{e\}\]
\[\gamma_0^\Sigma_2(D_4) = \mathbb{Z}/2\mathbb{Z}, \quad \gamma_1^\Sigma_2(D_4) = \langle p^2, s \rangle = D_2.\]

\section*{54 Schur's Theorem:}

We have a notion of equivalence of composition series.

Fix \(\Sigma_1: G = G_0 \geq \ldots \geq G_m = \{e\}\) and \(\Sigma_2: H = H_0 \geq \ldots \geq H_n = \{e\}\)

\textbf{Def.} We say \(\Sigma_1\) and \(\Sigma_2\) are equivalent if

(i) \(m = n\)

(ii) \(\exists \sigma \in S_n = \text{Aut}_{\text{set}}(\{0, \ldots, n-1\})\) such that \(\gamma_1^\Sigma_1(G) = \gamma_{\sigma(i)}^\Sigma_2(H).\)

\textbf{Ex.} \(G = \mathbb{Z}/4\mathbb{Z} \geq \mathbb{Z}/2\mathbb{Z} \geq \{e\}\) an equivalent \((\sigma = \text{id})\)

\(H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \geq \mathbb{Z}/2\mathbb{Z} \geq \{e\}\)

\textbf{Theorem (Schur)} Let \(\Sigma_1\) and \(\Sigma_2\) be two composition series of a group \(G\). Then, there exist composition series \(\Sigma_1'\) and \(\Sigma_2'\) finer than \(\Sigma_1\) and \(\Sigma_2\), respectively, such that \(\Sigma_1'\) and \(\Sigma_2'\) are equivalent.

\textbf{[Interpretation: any two composition series have a "common refinement", up to equivalence]}

\textbf{Obs!} Finest (simple graded pieces) + no repetitions as Jordan-Hölder series (Tomorrow)