Lecture 12: Short exact sequences, Composition Series

§1 Short exact sequences:

Recall A short exact sequence (ses) is a sequence of the form

$$\mathbb{1} \longrightarrow G_1 \xrightarrow{\varphi} G_2 \xrightarrow{\Psi} G_3 \longrightarrow \mathbb{1}$$

involving groups & group homomorphisms with

(i) φ injective (ii) $\operatorname{Ker}\Psi = \operatorname{Im}\varphi$ (iii) Ψ surjective.

Definition: A ses is split if we have a section, that is, a gp hom $s: G_3 \longrightarrow G_2$ with $\Psi \circ s = id_{G_3}$ ($\Rightarrow$ s is injective!)

Definition: A ses is trivial if we have a retraction (or projection), that is, a gp hom $r: G_2 \longrightarrow G_1$ with $r\circ\varphi = id_{G_1}$. ($\Rightarrow$ r is surjective!)

- See HW4 for examples. In particular.

Obs 1: Not every ses splits!

$\nexists s$ $\quad \mathbb{Z}/_{2\mathbb{Z}} \times \mathbb{Z}/_{2\mathbb{Z}}$

Ex: $\mathbb{1} \longrightarrow \{\pm1\} \longrightarrow Q_8 \longrightarrow Q_8/_{\{\pm1\}} \longrightarrow \mathbb{1}$.

only elements of order 2 $= \pm1$ $\quad$ all elements have order 2 (except $\{\pm1\}$)

Obs 2: trivial & split ses are different things!

Ex: $\mathbb{1} \longrightarrow A_3 \xrightarrow{i} S_3 \xrightarrow{\text{sign}} \{\pm1\} \longrightarrow \mathbb{1}$

$\nexists r$ gp hom $\quad \exists s$ gp hom

Claim 1: $s(-1) = (12)$ satisfies $\text{sign}\circ s = id_{\{\pm1\}}$. $\quad 1 \xmapsto{s} id \xrightarrow{\text{sign}} 1$, $(-1) \to (12) \to -1$.

($\Rightarrow$ ses splits)

Claim 2: $\nexists r: S_3 \longrightarrow A_3$ gp hom s.t. $r\circ i = id_{A_3}$ ($\Rightarrow$ ses not trivial!)

Why? Set $\sigma = r((ij))$ $(i\neq j)$ $o((ij)) = 2$ but $o(\sigma) \mid |A_3| = \frac{|S_3|}{2} = 3$ so $o(\sigma) = 1$.

But every permutation in S_3 is a product of transpositions so r must be trivial on S_3: $\operatorname{Im} r = \mathbb{1} \not\leq A_3$. Contr! since r is surjective $\square$

Lemma: A trivial ses always splits

Pf/
$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "\varphi"] & B \arrow[r, "\Psi"] \arrow[l, dashed, bend left, "\exists r"] & C \arrow[r] \arrow[l, dashed, bend left, "? \exists s"] & 1 \end{tikzcd}$$
A, B, C gps.

$r : B \twoheadrightarrow A \qquad r \circ \varphi = id_A$

Want to build a gp hom $s : C \hookrightarrow B$ with $\Psi \circ s = id_C$.

We write $\ker r \xrightarrow{\Psi|_{\ker}} C$ gp homomorphism.

Claim 1: $\Psi|_{\ker r}$ is injective.

Pf/ Pick $b \in \ker r$ with $\Psi(b) = e_C$. so $b \in \ker \Psi \overset{\text{exactness}}{=} \operatorname{Im} \varphi$

so $b = \varphi(a)$ for $a \in A$.

Then $e_A = r(b) = \underbrace{r \circ \varphi}_{1_A}(a) = a$ $\Bigg\} \Rightarrow b = \varphi(e_A) = e_B$ $\square$

Claim 2: $\Psi|_{\ker r}$ is surjective

Pf/ Given $c \in C$ pick $b \in B$ with $\Psi(b) = c$. This choice is not unique, but if $\Psi(b') = c$ then $b' = b\varphi(a)$ for $a \in A$.

Pick $b' = b\, \varphi \circ \underbrace{r(b^{-1})}_{\in A}$. Note: $b' \in \ker r$. because

$r(b') = r(b)\, \underbrace{r \circ \varphi}_{= 1_A} \circ r(b^{-1}) = r(b)\, r(b^{-1}) = e_B$. Also $\Psi(b') \overset{!}{=} \Psi(b)$

Then $\exists\ s : C \longrightarrow \ker r \hookrightarrow B$ gp homomorphism

with $\Psi \circ s = 1_C$. $\Longrightarrow$ the ses splits. $\square$

§2. Direct / Semidirect Products and s.e.s.:

Split & Trivial ses will characterize G_2 as $G_1 \rtimes_\alpha G_3$ or $G_1 \times G_3$

Proposition 1: If the ses $1 \to N \xrightarrow{\varphi} G \xrightarrow{\Psi} H \to 1$ is trivial, then

$G \simeq N \times H$ (direct product) when $N \xhookrightarrow{\varphi} G$ & $H \xhookrightarrow{s} G$

<u>Proof</u>: Assume $\exists\, r : G \longrightarrow N$ retraction. Then, by Lemma (page 2), we have a section $H \xhookrightarrow{s} G$

- The s.e.s. for the triple $(N, N\times H, H)$ is trivial.

We have

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r, "i"] \arrow[d, equal] & N\times H \arrow[r, "\pi_2"] \arrow[l, dashed, bend left, "\pi_1"] & H \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & N \arrow[r, "\varphi"] & G \arrow[r, "\psi"] \arrow[u, "??\ \cong"] \arrow[l, dashed, bend left, "\exists r"] & H \arrow[r] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & N\times H \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
N & & H
\end{tikzcd}$$

Define $\eta : G \longrightarrow N\times H$ via $\eta(g) = (r(g), \psi(g))$.

- η is gp hom since both r & ψ are

- <u>Claim 1</u>: η is surjective :

Pf/. Pick $x\in N$ & $h\in H$. Choose $g\in G$ with $\psi(g)=h$ ($\exists$ because ψ surj)

Take $\boxed{\tilde{g} = g\,(\varphi\circ r(g))^{-1}\,\varphi(x) \in G}$

$$\Rightarrow \psi(\tilde{g}) = \psi(g)\,\psi(\varphi\circ r(g^{-1}))\,\underbrace{\psi\circ\varphi(x)}_{=e_H} = \psi(g)\,\underbrace{\psi\circ\varphi(\underbrace{r(g^{-1})}_{\in N})}_{=e_H} = \psi(g) = h$$

$$r(\tilde{g}) = r(g)\,\underbrace{r(\varphi}_{id_N}\circ r(g^{-1}))\,\underbrace{r\circ\varphi}_{id_N}(x) = \underbrace{r(g)\,r(g^{-1})}_{=e_G}\cdot x = x \in N$$ □

So $\eta(\tilde{g}) = (x,h)$

- <u>Claim 2</u>: η is injective.

Pf/ If $\eta(g) = (e_N, e_H)$ then $\Psi(g) = e_H$, so $g\in \operatorname{Ker}\Psi = \operatorname{Im}\varphi$.

$$\left.\begin{array}{l} \text{Then, } \exists\ x\in N \text{ with } g = \varphi(x) \\ \Rightarrow e_N = r(g) = r\circ\varphi(x) = x \end{array}\right\} \Rightarrow g = \varphi(e_N) = e_G.$$

It is easy to check all squares commute. □

<u>Proposition 2</u>: If a ses $1 \to N \xrightarrow{\varphi} G \xrightarrow{\Psi} H \to 1$ splits, then $G \simeq N \rtimes H$ where $N \xhookrightarrow{\varphi} G$ & $H \xhookrightarrow{s} G$

Pf/ Know: $N \underset{\varphi}{\trianglelefteq} G$ & $H \underset{s}{<} G$.

Claim 1: $s(H) \cap \varphi(N) = \{e\}$

Pick $g \in s(H) \cap \varphi(N)$ then $g = s(h) = \varphi(x)$ $x \in N, h \in H$

$$\left.\begin{aligned} \Rightarrow \Psi(g) &= \Psi \circ s(h) = h \\ &= \Psi \circ \varphi(x) = e_H \end{aligned}\right\} \Rightarrow g = s(e_H) = e_G \checkmark$$

Claim 2: $NH = \{\varphi(x)\, s(h) \quad x \in N, h \in H\} = G$

Pick $g \in G \Rightarrow \Psi(g) \in H$

Pick $\tilde{g} = (s \circ \Psi)(g)$. It satisfies $\Psi(g) = \Psi(\tilde{g})$, so

$\tilde{g}^{-1} g \in \operatorname{Ker} \Psi = \operatorname{Im} \varphi$ so $\tilde{g}^{-1} g = \varphi(x)$ for some $x \in N$.

Then: $g = \tilde{g}\, \varphi(x) = s \circ \Psi(g)\, \varphi(x) = \underbrace{s \circ \Psi(g)\, \underset{N \trianglelefteq G}{\varphi(x)}\, (s \circ \Psi(g))^{-1}}_{\in N} \underbrace{(s \circ \Psi)(g)}_{\in H}$

By definition, $G = \varphi(N) \rtimes s(H) \simeq N \rtimes H$. $\square$

Example (HW 4): $1 \longrightarrow A_n \xrightarrow{i} S_n \xrightarrow{\text{sign}} \{\pm 1\} \longrightarrow 1$ splits

$(12) \longleftarrow -1$

so $S_n = A_n \rtimes \mathbb{Z}/2\mathbb{Z}$.

§3 Composition Series

Recall: A group S is called simple if $\{e\}$ & S are the only normal subgroups of S

Examples: A_n $n \geq 5$ are simple (next week)

$\mathbb{Z}/p\mathbb{Z}$ $p > 0$ prime are simple

$PSL_n = SL_n / Z(SL_n)$ are simple

Def: A <u>composition series</u> of a group G is a finite sequence of subgroups of G

$$\Sigma: \quad G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_k = \{e\}$$

such that $G_{j+1} \triangleleft G_j$ is normal for all $j = 0, \dots, k-1$.

The successive quotients: $gr_i(G) := G_i / G_{i+1}$ $\quad 0 \leq i \leq k-1$.

(Other notation: $gr_i^{\Sigma}(G)$ if Σ is not clear from context.)

Def 2. A composition series Σ' is said to refine Σ (or be finer than Σ) if Σ is obtained from Σ' by omitting some terms.

More precisely: $\Sigma': \ G = G'_0 \supseteq \cdots \supseteq G'_m = \{e\}$

$\Sigma: \ G = G_0 \supseteq \cdots \supseteq G_n = \{e\}$

Σ' is finer than Σ if $n \leq m$ and there exists an order-preserving injective map $\Phi: \{0, \dots, n\} \longrightarrow \{0, \dots, m\}$ with $G_j = G'_{\phi(j)}$ $\forall j$.

Ex 1 $\Sigma_1: G = \mathbb{Z}/6\mathbb{Z} \supseteq \mathbb{Z}/3\mathbb{Z} \supseteq \{e\}$ $\quad$ no refinement, only 1 coarsening

$\Sigma_2: G = \mathbb{Z}/6\mathbb{Z} \supseteq \mathbb{Z}/2\mathbb{Z} \supseteq \{e\}$

$gr_0^{\Sigma_1}(G) = \mathbb{Z}/6\mathbb{Z} / \mathbb{Z}/3\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} = gr_1^{\Sigma_2}(G)$, $\quad gr_1^{\Sigma_1} = \mathbb{Z}/3\mathbb{Z} = gr_0^{\Sigma_2}(G)$.

$\Sigma_0: \ \mathbb{Z}/6\mathbb{Z} \supseteq \{e\}$ $\quad$, Σ_1 refines Σ via $\phi: \{0,1\} \to \{0,1,2\}$ $\ \phi(0) = 0, \phi(1) = 2$

Remark: In general, a series obtained from a composition series Σ' by omitting some terms is <u>NOT</u> a composition series since for $j > i+1$, G'_j is not in general a normal subgroup of G'_i.

Ex 1: $G = D_4 \supseteq \langle \rho^2, s \rangle \supseteq \langle s \rangle \supseteq \{e\}$ $\quad \left(\begin{matrix} s\rho^2 = \rho^{-2}s \\ s^{-1} = s, \ \rho^4 = 1 \end{matrix} \right)$

$G_1 = \langle \rho^2, s \rangle \triangleleft G$ $\quad \rho\rho^2\rho^{-1} = \rho^2 \in G_1$ $\quad \rho s \rho^{-1} = \rho^2 s \in G_1$

$s\rho^2 s^{-1} = \rho^2 \in G_1$ $\quad s s s^{-1} = s \in G_1$

$G_2 = \langle s \rangle \triangleleft \langle \rho^2, s \rangle$ $\quad \rho^2 s \rho^{-2} = \rho^2 \rho^2 s = \rho^4 s = s \in G_1$

$s s s^{-1} = s \in G_1$

$G_3 = \{e\} \triangleleft \langle s \rangle$

$$gr_0(D_4) = \frac{\langle \rho, s\rangle}{\langle \rho^2, s\rangle} \cong \frac{\langle \rho \rangle}{\langle \rho^2\rangle} \simeq \mathbb{Z}/2\mathbb{Z}$$

$$gr_1(D_4) = \frac{\langle \rho^2, s\rangle}{\langle s\rangle} \cong \langle \rho^2 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$$

$$gr_2(D_4) = \frac{\langle s\rangle}{\langle e\rangle} \simeq \mathbb{Z}/2\mathbb{Z}$$

- We can't omit $\langle \rho^2 s\rangle$ and have a composition series because $\langle s\rangle \ntrianglelefteq D_4$.
- We can omit $\langle s \rangle$ and get a comp series

$$\Sigma_2 : \quad D_4 \supseteq \langle \rho^2, s\rangle \supseteq \{e\}$$

$$gr_0^{\Sigma_2}(D_4) \cong \mathbb{Z}/2\mathbb{Z}, \quad gr_1^{\Sigma_2}(D_4) = \langle \rho^2, s\rangle = D_2 .$$

§4 Schrier's Theorem:

We have a notion of equivalence of composition series

Fix
$$\Sigma_1 : \quad G = G_0 \supseteq \dots \supseteq G_m = \{e\}$$
$$\Sigma_2 : \quad H = H_0 \supseteq \dots \supseteq H_n = \{e\}$$
two composition series

Def: We say Σ_1 & Σ_2 are equivalent if

(i) $m = n$

(ii) $\exists\ \sigma \in S_n = \mathrm{Aut}_{Set}(\{0, \dots, n-1\})$ such that $gr_i^{\Sigma_1}(G) = gr_{\sigma(i)}^{\Sigma_2}(H)$. $\forall i$

Ex:
$$G = \mathbb{Z}/4\mathbb{Z} \supseteq \mathbb{Z}/2\mathbb{Z} \supseteq \{e\}$$
$$H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \supseteq \mathbb{Z}/2\mathbb{Z} \supseteq \{e\}$$
are equivalent ($\sigma = id$)

Theorem (Schrier) Let Σ_1 & Σ_2 be two composition series of a group G. Then, there exist composition series Σ_1' & Σ_2' finer than Σ_1 & Σ_2, respectively such that Σ_1' & Σ_2' are equivalent.

[Interpretation: any two composition series have a "common refinement", up to equivalence]

- **Obs**: Finest (= simple graded pieces) + no repetitions $\rightsquigarrow$ Jordan–Hölder series (TOMORROW)