Lecture 13: Zassenhaus Lemma, Jordan-Hölder Series

Last time: Discussed composition series

- A composition series of a group G is a finite sequence of subgroups of G

$$\Sigma: \quad G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_k = \{e\}$$

such that $G_{j+1} \trianglelefteq G_j$ is normal for all $j = 0, \ldots, k-1$.

- Graded pieces: $gr_i(G) := G_i / G_{i+1}$ $\quad 0 \leq i \leq k-1$.
- Refinement: add terms to the composition series while remaining one
- Equivalence:
  - Same number of terms
  - — graded pieces, counted with multiplicity (up to permutation) (up to isomorphism)

TODAY:
- Proof of Schrier's Theorem
- Discuss maximally refined comp. series = Jordan Hölder series.

§1 Schrier's Theorem & Zassenhaus Lemma.

Theorem (Schrier) Any two composition series of a group have a "common refinement", up to equivalence.

Ex: $G = \mathbb{Z}/6\mathbb{Z}$

$$\Sigma_1 = \Sigma_1' : \mathbb{Z}/6\mathbb{Z} \supseteq \mathbb{Z}/3\mathbb{Z} \supseteq \{e\}$$

$$\Sigma_2 = \Sigma_2' : \mathbb{Z}/6\mathbb{Z} \supseteq \mathbb{Z}/2\mathbb{Z} \supseteq \{e\}$$

graded pieces: $\mathbb{Z}/2\mathbb{Z}$ & $\mathbb{Z}/3\mathbb{Z}$ $\qquad$ Σ_1 & Σ_2 are equivalent via

$\sigma \in S_2 : 0 \mapsto 1, \; 1 \mapsto 0 \qquad gr_0^{\Sigma_1} = gr_1^{\Sigma_2}, \; gr_1^{\Sigma_1} = gr_0^{\Sigma_2}$.

Proof: Write $\Sigma_1: G = H_0 \supseteq \cdots \supseteq H_n = \{e\}$

$\Sigma_2: G = K_0 \supseteq \cdots \supseteq K_p = \{e\}$

Idea: ① For each $i = 0, \ldots, n-1$, use Σ_2 to insert $(p-1)$ many groups $\{H'_{ij}\}_{j=1}^{p-1}$ in between H_i & H_{i+1} $\rightsquigarrow$ get Σ_1' finer than Σ_1

② Similarly, use Σ_1 to insert $(n-1)$ many subgroups $\{K'_{j,i}\}_{i=1}^{n-1}$ in between K_j & K_{j+1} $\rightsquigarrow$ get Σ_2' finer than Σ_2.

③ Show Σ_1' & Σ_2' are equivalent.

Define: $i=0,\dots,n$, $j=0,\dots,p$, $\boxed{H'_{i,j} := H_{i+1}(H_i \cap K_j)}$ & $\boxed{K'_{j,i} := K_{j+1}(H_i \cap K_j)}$

It is clear that: $H'_{i,0} = H_i$, $H'_{i,p} = H_{i+1}$; $K'_{j,0} = K_j$, $K'_{j,n} = K_{j+1}$.

- $H'_{i,j+1} < H'_{i,j}$; $K'_{j,i+1} < K'_{j,i}$ $\forall i,j$ are subgroups by 3rd Iso theorem ($H_i \cap K_j < H_i$, $H_{i+1} \trianglelefteq H_i \Rightarrow H'_{i,j} < H_i$) & $H'_{i,j+1} \subseteq H'_{i,j}$, $K'_{j,i+1} \subseteq K'_{j,i}$)

We need to check these subgroups are normal!

Claim: (i) $H'_{i,j+1} \triangleleft H'_{i,j}$, $K'_{j,i+1} \triangleleft K'_{j,i}$

(ii) $H'_{i,j}/H'_{i,j+1} \simeq K'_{j,i}/K'_{j,i+1}$

To simplify notation, write $H = H_i \triangleright H' = H_{i+1}$

$K = K_j \triangleright K' = K_{j+1}$

The claims will follow using Zassenhaus' Lemma $\square$

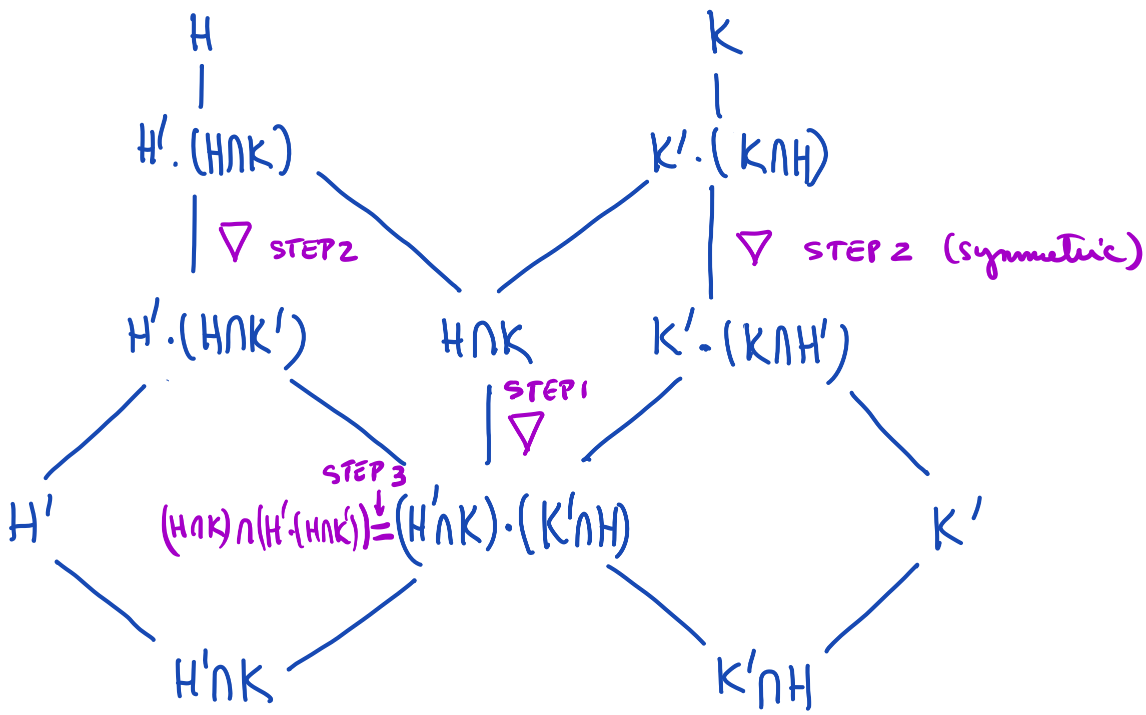
Lemma (Zassenhaus). Fix a group G, H, K two subgroups of G & $H' \triangleleft H$, $K' \triangleleft K$. Then:

(i) $H'\cdot(H\cap K') \triangleleft H'\cdot(H\cap K)$

$K'\cdot(H'\cap K) \triangleleft K'\cdot(H\cap K)$

(ii) $\dfrac{H'\cdot(H\cap K)}{H'\cdot(H\cap K')} \simeq \dfrac{K'\cdot(H\cap K)}{K'\cdot(H'\cap K)}$

Proof: The next picture outlines the proof:



STEP 1 $(H' \cap K) \cdot (K' \cap H) \triangleleft H \cap K$

This is true because $\left.\begin{array}{lll} H' \triangleleft H & \text{so} & H' \cap K \triangleleft H \cap K \\ K' \triangleleft K & \text{so} & K' \cap H \triangleleft H \cap K \end{array}\right\} \Rightarrow$ By 3^{rd} Iso

$(H' \cap K)(K' \cap H) < H \cap K$ (use $H' \cap K \triangleleft H \cap K$, $K' \cap H < H \cap K$.

But $g(H' \cap K)\, K' \cap H\, g^{-1} = g(H' \cap K) g^{-1} g (K' \cap H) g^{-1} \subseteq (H' \cap K)(K' \cap H)$ for $g \in H \cap K$

so $(H' \cap K)(K' \cap H) \triangleleft H \cap K$.

STEP 2: $H'(H \cap K') \triangleleft H'(H \cap K)$

This follows from a more general statement:

Lemma: If $\mathcal{G}$ is a group, $G_1 < \mathcal{G}$, $N \triangleleft \mathcal{G}$ & $G_2 \triangleleft G_1$, then $N \cdot G_2 \triangleleft N \cdot G_1$

PF/ $G_2 \cdot N = N \cdot G_2 < \mathcal{G}$ & $G_1 N = N G_1 < \mathcal{G}$ by 3^{rd} Isomorphism Thm.

Then $x = (g_1 n) n' g_2 (g_1 n)^{-1} = g_1 n n' g_2 n^{-1} g_1^{-1} = \underbrace{(g_1 n g_1^{-1})}_{\in N} \underbrace{(g_1 n' g_1^{-1})}_{\in N} \underbrace{(g_1 g_2 g_1^{-1})}_{\in G_2} \underbrace{(g_1 n^{-1} g^{-1})}_{\in N}$

$\forall g_1 \in G_1$, $n, n' \in N$ $g_2 \in G_2$.

$\Rightarrow x \in N N G_2 N = N G_2 N = N N G_2 = N G_2$ so $G_2 N \triangleleft G_1 N$ $\square$

(*) **Alternative proof of the Lemma**:

Let $\pi : \mathcal{G} \longrightarrow \mathcal{G}/N$ be the natural projection & $\overline{G_1} = \pi(G_1)$

Restrict π to G_1 to get $\pi : G_1 \twoheadrightarrow \overline{G_1}$, hence $\overline{G_2} := \pi(G_2)$ is a normal subgroup of $\overline{G_1}$ (because π is surjective).

Now, consider the homomorphism defined by:

$$G_1 \cdot N \hookrightarrow \mathcal{G} \longrightarrow \mathcal{G}/N$$

$$\alpha : G_1 N \longrightarrow \overline{G_1}$$

Then $\alpha^{-1}(\overline{G_2}) = G_2 \cdot N$ is the inverse image of a normal subgroup, hence normal. $\square$

Then STEP 2 follows by taking $\mathcal{G} = H$, $N = H'$, $G_1 = H \cap K$, $G_2 = H \cap K'$.

STEP 3: Use the 3$^{\text{rd}}$ Isomorphism Theorem:

$$\frac{H'(H\cap K)}{H'(H\cap K')} \simeq \frac{H\cap K}{(H\cap K)\cap(H'\cdot(H\cap K'))} \qquad \begin{cases} N = H'\cdot(H\cap K') \trianglelefteq H \\ \tilde{H} = H\cap K \end{cases}$$

Claim: $(H\cap K)\cap(H'\cdot(H\cap K')) = (H'\cap K)\cdot(K'\cap H)$

Pf/ $(H'\cap K)(K'\cap H) \subset (H\cap K)\cap(H'\cdot(H\cap K'))$ is clear.

Conversely, let $x = a\cdot b \in H'\cdot(H\cap K')\cap(H\cap K)$ with $a\in H'$, $b\in H\cap K' \subset H\cap K$

$\Rightarrow a = x b^{-1} \in (H\cap K)\cdot(H\cap K) \subseteq H\cap K$

$\Rightarrow a \in H'\cap(H\cap K) = H'\cap K$.

Thus, $x = ab \in (H'\cap K)(H\cap K')$. $\square$

Swapping the roles of H & K, H' & K', combined with the Claim we get

$$\frac{K'(H\cap K)}{K'(H\cap K')} \simeq \frac{H\cap K}{(H\cap K')(K\cap H')} \simeq \frac{H'(H\cap K)}{H'(H\cap K')} \qquad \square$$

§2 Jordan-Hölder Series

Definition A composition series Σ: $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{e\}$ is said to be a Jordan-Hölder series if:

(i) Σ is strictly decreasing (ie $G_{j+1} \subsetneq G_j$ $\forall j = 0,\dots,n-1$)

(ii) There is no strictly decreasing composition series distinct from Σ and finer than Σ.

Proposition: A composition series Σ of G is Jordan-Hölder (or JH for short) if and only if $gr_i^{\Sigma}(G)$ is simple for all $i = 0,\dots,n-1$.

(Recall: $\{e\}$ is not simple; G is simple if $H \trianglelefteq G \Rightarrow H = \{e\}$ or G)

Proof: Note that a composition series is strictly decreasing if and only if none of its associated quotients is $\{e\}$.

Let $\Sigma: G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_n = \{e\}$ be a strictly decreasing composition series that is not JH. Then, there exists a strictly decreasing series Σ' finer than Σ. Thus, we can find $i = 0, \dots, n-1$ where $G_{i+1} \ntrianglelefteq G_i$ are not consecutive in Σ'. That is, there exist intermediary normal subgroups:

$$G_{i+1} \ntrianglelefteq H_k \trianglelefteq \cdots\cdot H_2 \ntrianglelefteq H_1 \ntrianglelefteq G_i$$

In particular, $G_{i+1} \triangleleft H_1$ since $G_{i+1} \triangleleft G_i$ & $G_{i+1} < H_1 < G_i$.

Hence, H_1/G_{i+1} is a nontrivial normal subgroup of G_i/G_{i+1}, so $gr_i(G)$ is not simple.

Conversely, assume $\Sigma: G = G_0 \supsetneq \cdots \supsetneq G_n = \{e\}$ is a strictly decreasing composition series, one of whose graded pieces, say G_i/G_{i+1} is not simple. By the second Isomorphism Theorem, a proper, nontrivial normal subgroup of G_i/G_{i+1} is of the form H/G_{i+1} for some intermediate normal subgroup

$$G_{i+1} < H \triangleleft G_i \qquad . \text{ Thus, } G_{i+1} \ntrianglelefteq H \ntrianglelefteq G_{i+1})$$

Conclude: $\Sigma': G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_i \supsetneq H \supsetneq G_{i+1} \supsetneq \cdots \supsetneq G_n = \{e\}$ is finer than Σ, so Σ is not J-H. $\square$

⚠ A general group G need <u>NOT</u> posses a JH series

<u>Ex.</u> $\mathbb{Z} \supsetneq 2\mathbb{Z} \supsetneq 4\mathbb{Z} \supsetneq 8\mathbb{Z} \supsetneq \cdots \supsetneq G_k = 2^k\mathbb{Z} \supsetneq \cdots$ cannot terminate

However, every <u>finite</u> group G has a Jordan-Hölder (By induction on $|G|$)

More precisely, pick H_1 maximal among all proper, normal subgroups of G, recursively let H_{n+1} be maximal among proper normal subgroups of H_n. This procedure must halt (in, at most, $|G|$ steps), thus forming a JH series)

<u>Theorem</u> (Jordan-Hölder) Two Jordan-Hölder series of a group G are equivalent.

<u>Proof</u>: Let Σ_1, Σ_2 be two JH series of G. By Schrier's Thm, we can refine them to Σ_1' & Σ_2' where Σ_1' & Σ_2' are equivalent.

As Σ_1 (and Σ_2) is JH, Σ_1' (and Σ_2') is either identical to Σ_1 (resp. Σ_2) or it is obtained from Σ_1 (resp. Σ_2) by repeating some terms. As the series of quotients of Σ_1' & Σ_2' differ only in the order of the graded pieces, after removing all trivial quotients, the same is true for Σ_1 & Σ_2 □

Ex: $G = \mathbb{Z}/6\mathbb{Z}$

$$\Sigma_1 : \mathbb{Z}/6\mathbb{Z} \supsetneq \mathbb{Z}/3\mathbb{Z} \supsetneq \{e\} \quad \text{JH}$$
$$\Sigma_2 : \mathbb{Z}/6\mathbb{Z} \supsetneq \mathbb{Z}/2\mathbb{Z} \supsetneq \{e\} \quad \text{JH}$$

graded pieces: $gr_0^{\Sigma_1}(G) = \mathbb{Z}/2\mathbb{Z} = gr_1^{\Sigma_2}(G)$

$gr_1^{\Sigma_1}(G) = \mathbb{Z}/3\mathbb{Z} = gr_0^{\Sigma_2}(G)$

Corollary: Let G be a group that admits a JH series. If Σ is any strictly decreasing composition series of G, then there exists a JH series refining Σ.

Sketch of a proof: Let Σ_0 be a J-H series of G. By Schreier's Thm, we can find Σ_0' & Σ' two equivalent composition series refining Σ_0 & Σ, resp. The proof of JH Theorem ensures that Σ_0' is JH & so Σ' is also JH.

Example 1: $G = \mathbb{Z}/p^k\mathbb{Z} = \langle g \rangle$ $\quad k>1$ $\quad$ graded pieces for JH $= \mathbb{Z}/p\mathbb{Z}$. (simple & order $| p^k$)

$\Sigma: G = G_0 \geq G_1 = \mathbb{Z}/p^{k-1}\mathbb{Z} \geq G_2 = \mathbb{Z}/p^{k-2}\mathbb{Z} \geq \cdots \geq G_{k-1} = \mathbb{Z}/p\mathbb{Z} \geq G_k = \{e\}$ is JH.

$G_1 = \langle g^p \rangle$, $G_2 = \langle g^{p^2} \rangle$, $G_{k-1} = \langle g^{p^{k-1}} \rangle$

Example 2: $G = \mathbb{Z}/n\mathbb{Z}$ $\quad$ How to build a JH series for G?

- If n is prime, G is simple so $G \geq \{e\}$ is JH
- If n is not prime, write $n = p_1^{a_1} \cdots p_r^{a_r}$ prime decomposition.

$$\Rightarrow G = \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/\tfrac{n}{p_1^{a_1}}\mathbb{Z} = \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \left(\mathbb{Z}/p_2^{a_2}\mathbb{Z} \times \mathbb{Z}/\tfrac{n}{p_1^{a_1}p_2^{a_2}}\mathbb{Z} \right) = \cdots$$
$$= \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_r^{a_r}\mathbb{Z}$$

$$\Rightarrow G = G_0 \supsetneq G_1 := \mathbb{Z}/_{\frac{n}{p_1^{a_1}}\mathbb{Z}} \geq G_2 := \mathbb{Z}/_{\frac{n}{p_1^{a_1} p_2^{a_2}}\mathbb{Z}} \geq \cdots \geq G_{r-1} := \mathbb{Z}/_{p_r^{a_r}\mathbb{Z}} \geq G_r = \{e\}$$

comp series with graded pieces = p-groups.

We can refine each $G_i = \mathbb{Z}/_{\frac{n}{p_1^{a_1}\cdots p_i^{a_i}}\mathbb{Z}} \geq G_{i+1} = \mathbb{Z}/_{\frac{n}{p_1^{a_1}\cdots p_{i+1}^{a_{i+1}}}\mathbb{Z}}$

by lifting a JH series of $\mathbb{Z}/_{p_{i+1}^{a_{i+1}}\mathbb{Z}}$ (use Example 1)