Lecture 13: Zassenhaus Lemma, Jordan-Hölder Series

Last time: Discussed composition series

1. A composition series of a group \( G \) is a finite sequence of subgroups of \( G \)
   \[ \Sigma : \quad G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_k = \{ e \} \]
   such that \( G_{j+1} \triangleleft G_j \) is normal for all \( j = 0, \ldots, k-1 \).

2. Graded pieces: \( \varphi_i (G) := \frac{G_i}{G_{i+1}} \quad 0 \leq i \leq k-1 \).

3. Refinement: add terms to the composition series while remaining the same number of terms.

4. Equivalence: graded pieces, counted with multiplicity (up to permutation) (up to isomorphism).

Today: Proof of Schrier’s Theorem

Discuss maximally refined comp. series \( \Rightarrow \) Jordan-Hölder series.

Schrier’s Theorem & Zassenhaus Lemma:

**Theorem (Schrier)** Any two composition series of a group have a “common refinement” up to equivalence.

**Example:** \( G = \mathbb{Z}/6\mathbb{Z} \)

\[ \Sigma_1 : \quad \mathbb{Z}/6\mathbb{Z} \supseteq \mathbb{Z}/3\mathbb{Z} \supseteq \{ e \} \]

\[ \Sigma_2 : \quad \mathbb{Z}/6\mathbb{Z} \supseteq \mathbb{Z}/2\mathbb{Z} \supseteq \{ e \} \]

Graded pieces: \( \mathbb{Z}/2\mathbb{Z} \) & \( \mathbb{Z}/3\mathbb{Z} \) \( \Sigma_1 \) & \( \Sigma_2 \) are equivalent via \( \sigma \in S_2 : \quad 0 \mapsto 1 \), \( \varphi_1 \Sigma_1 = \varphi_1 \Sigma_2 \)

**Proof:** Write \( \Sigma_1 : \quad G = H_0 \supsetneq \cdots \supsetneq H_n = \{ e \} \)

\[ \Sigma_2 : \quad G = K_0 \supsetneq \cdots \supsetneq K_p = \{ e \} \]

**Idea:**

1. For each \( i = 0, \ldots, n-1 \) use \( \Sigma_2 \) to insert \( (n-1) \) many groups \( \{ H'_{ij} \}_{j=1}^{n-1} \) in between \( H_i \) & \( H_{i+1} \).

2. Similarly, use \( \Sigma_1 \) to insert \( (n-1) \) many subgroups \( \{ K'_{ij} \}_{j=1}^{n-1} \) in between \( K_i \) & \( K_{i+1} \).

3. Show \( \Sigma_1' \) & \( \Sigma_2' \) are equivalent.
Define: \[ H'_{ij} := H'_{i+1} (H_i \cap K_j) \quad \& \quad K'_{ij} := K'_{j+1} (H_i \cap K_j) \]

It is clear that: \( H'_{i,0} = H_i, \quad H'_{i,n} = H_i + 1, \quad K'_{j,0} = K_j, \quad K'_{j,n} = K_j + 1. \)

\( H'_{i,j+1} < H'_{i,j} \); \( K'_{i,j+1} < K'_{i,j} \); \( H_{i,j} \) are subgroups by 3rd Iso Theorem \((H_i \cap K_j < H_i, H_i + H_i \Rightarrow H_{i,j} < H_i) \) \& \( H'_{i,j+1} \leq H'_{i,j} \)

We need to check these subgroups are normal!

Claim: (i) \( H'_{i,j+1} \triangleleft H'_{i,j} \), \( K'_{j,i+1} \triangleleft K'_{j,i} \)

(ii) \( H'_{i,j} / H'_{i,j+1} \cong K'_{j,i} / K'_{j,i+1} \)

To simplify notation, write \( H = H_i \triangleright H' = H_{i+1} \), \( K = K_j \triangleright K' = K_{j+1} \)

The claims will follow using Zassenhaus' Lemma.

Lemma (Zassenhaus): Fix a group \( G \). \( H, K \) two subgroups of \( G \) \& \( H' \triangleleft H \), \( K' \triangleleft K \). Then:

(i) \( H' \cdot (H \cap K') \triangleleft H' \cdot (H \\cap K) \)

\( K' \cdot (H' \cap K) \triangleleft K' \cdot (H \cap K) \)

(ii) \( H' \cdot (H \cap K) \cong K' \cdot (H \cap K) \)

Proof: The next picture outlines the proof:

\[
\begin{align*}
\text{STEP 1} & \quad H' \cap (H \cap K) = \text{(symmetric)} \\
\text{STEP 2} & \quad H' \cdot (H' \cap K') = H' \cdot (H' \cap K) \\
\text{STEP 3} & \quad (H \cap K) \cap (H' \cap K') = (H' \cap K) \cdot (K' \cap H)
\end{align*}
\]
STEP 1: \((H' \cap K')(K' \cap H') \triangleleft H \cap K\)

This is true because \(H' \triangleleft H\) so \(H' \cap K \triangleleft H \cap K\) \(\Rightarrow \) by 3rd Isomorphism Theorem

\((H' \cap K')(K' \cap H') \subseteq H \cap K\) (since \(H' \cap K \cap H' \subseteq H \cap K\)).

But \(g(H' \cap K')K' \cap H' g^{-1} = g(H' \cap K)g^{-1}g(K' \cap H)g^{-1} = (H' \cap K)(K' \cap H) \cap gH \cap K\)

so \((H' \cap K')(K' \cap H') \triangleleft H \cap K\).

STEP 2: \(H'(H' \cap K') \triangleleft H'(H' \cap K')\)

This follows from a more general statement:

**Lemma:** If \(G\) is a group, \(G_1 \triangleleft G\), \(N \triangleleft G\) and \(G_2 \triangleleft G_1\), then \(N, G_2 \triangleleft N \cdot G_1\).

3rd Isomorphism Theorem:

Then \(x = g_1, w_1 \in G_2\) \(\Rightarrow (s, n) \triangleleft (s, n) = G_2 \cdot N = N \cdot G_2 \triangleleft G\) by 3rd Isomorphism Theorem.

Then \(x = g_1, w_1 \in G_2\) \(\Rightarrow (s, n) \triangleleft (s, n) = G_2 \cdot N = N \cdot G_2 \triangleleft G\) by 3rd Isomorphism Theorem.

\[x \in N \cap G_2 \Rightarrow x \in N \cdot G_2 \triangleleft G_2 N \triangleleft G_1 N \triangleleft G_1\]

\[(#) \text{Alternative proof of the Lemma:}
\]

Let \(\overline{\pi}: G \rightarrow G/N\) be the natural projection \(\overline{G_1} = \overline{\pi}(G_1)\)

Restrict \(\pi\) to \(G_1\) to get \(\overline{\pi}: G_1 \rightarrow \overline{G_1}\), hence \(\overline{G_2} = \overline{\pi}(G_2)\)

is a normal subgroup of \(\overline{G_1}\) (because \(\pi\) is surjective).

Now, consider the homomorphism defined by:

\[G_1 \cdot N \rightarrow G \rightarrow G/N\]

\[\alpha: G_1 \cdot N \rightarrow \overline{G_1}\]

Then \(\alpha^{-1}(\overline{G_2}) = G_2 \cdot N\) is the inverse image of a normal subgroup, hence normal.

Then STEP 2 follows by taking \(G = H\), \(N = H'\), \(G_1 = H \cap K\), \(G_2 = H' \cap K\).
**STEP 3:** Use the 3rd Isomorphism Theorem:

\[
\frac{H'(HK)}{H(HK')} \cong \frac{HK}{H(HK') \cap (H'(HK'))} \quad \left\{ \begin{array}{l}
N = H'(HK') \triangleleft H \\
\tilde{H} = HK
\end{array} \right.
\]

**Claim:** \((HK \cap (H'(HK')) = (H' \cap HK) \cdot (H' \cap HK))

Proof: \((H' \cap HK)(H' \cap HK) \subseteq \text{clear}

Conversely, let \(x = a \cdot b \in H'(HK') \cap HK\) with \(a \in H', b \in HK'\)

\[\Rightarrow a = x \cdot b^{-1} \in (HK) \cdot (HK) \subseteq HK\]

\[\Rightarrow a \in H' \cap HK = H' \cap HK.\]

Thus, \(x = a \cdot b \in (H' \cap HK)(HK').\)

Swapping the roles of \(H \cap HK, H' \cap HK'\), combined with the claim we get

\[
\frac{K'(HK)}{K(HK')} \cong \frac{HK}{(HK')(HK')} \cong \frac{H'(HK)}{H'(HK')} \qed
\]

### §2 Jordan-Hölder Series

**Definition:** A composition series \(\Sigma: G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_n = \text{trivial}\)

is said to be a Jordan-Hölder series if:

(i) \(\Sigma\) is strictly decreasing (i.e. \(G_{j+1} \supseteq G_j\) \(\forall j = 0, \ldots, n-1\))

(ii) There is no strictly decreasing composition series distinct from \(\Sigma\) and finer than \(\Sigma\).

**Proposition:** A composition series \(\Sigma\) of \(G\) is Jordan-Hölder (or JH for short) if and only if \(\gamma_i(G)\) is simple for all \(i = 0, \ldots, n-1\).

(Recall: \(\text{trivial}\) is not simple; \(G\) is simple if \(H \triangleleft G \Rightarrow H = \{e\}\) or \(G\))

**Proof:** Note that a composition series is strictly decreasing if and only if none of its associated quotients is trivial.
Let \( \Sigma : G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_n = \{e\} \) be a strictly decreasing composition series that is not JH. Then, there exists a strictly decreasing series \( \Sigma' \) finer than \( \Sigma \). Thus, we can find \( i = 0, \ldots, n-1 \) where \( G_{i+1} \not\supset G_i \) are not consecutive in \( \Sigma' \). That is, there exist intermediary normal subgroups:
\[
G_{i+1} \not\supset H_k \supset \cdots \supset H_2 \supset H_1 \supset G_{i+1}
\]
In particular, \( G_{i+1} \not\supset H_1 \), since \( G_{i+1} \not\supset G_i \) & \( G_{i+1} \not\subset H_1 \subset G_i \).

Hence, \( H_1 / G_{i+1} \) is a nontrivial normal subgroup of \( G_i / G_{i+1} \), so \( G_i(G) \) is not simple.

Conversely, assume \( \Sigma : G = G_0 \supsetneq \cdots \supsetneq G_n = \{e\} \) is a strictly decreasing composition series, one of whose graded pieces, say \( G_i / G_{i+1} \), is not simple. By the second Isomorphism Theorem, a proper, nontrivial normal subgroup of \( G_i / G_{i+1} \) is of the form \( H / G_{i+1} \) for some intermediate normal subgroup
\[
G_{i+1} \not\supset H \supset G_i.
\]
Conclude: \( \Sigma' : G = G_0 \supsetneq G_1 \supsetneq \cdots \supset G_i \supsetneq H \supset G_{i+1} \supset \cdots \supset G_n = \{e\} \) is finer than \( \Sigma \), so \( \Sigma \) is not J-H. \( \Box \)

⚠️ A general group \( G \) need not posses a JH series,

\[
\text{Ex. } \mathbb{Z} \supset \mathbb{Z} / 2 \mathbb{Z} \supset \mathbb{Z} / 4 \mathbb{Z} \supset \cdots \supset \mathbb{Z} / 2^k \mathbb{Z} \supset \cdots \text{ cannot terminate}
\]

However, every finite group \( G \) has a Jordan–Hölder (By induction on \( |G| \))

More precisely, pick \( H \), maximal among all proper, normal subgroups of \( G \), recursively let \( H_{n+1} \) be maximal among proper normal subgroups of \( H_n \). This procedure must halt (in, at most, \( |G| \) steps), thus forming a JH series.

**Theorem (Jordan–Hölder)** Two Jordan–Hölder series of a group \( G \) are equivalent.

**Proof:** Let \( \Sigma_1, \Sigma_2 \) be two JH series of \( G \). By Schier's Thm, we can refine them to \( \Sigma'_1 \) & \( \Sigma'_2 \) where \( \Sigma'_1 \) & \( \Sigma'_2 \) are equivalent.
As $\Sigma_1$ (and $\Sigma_2$) is JH, $\Sigma'_1$ (and $\Sigma'_2$) is either identical to $\Sigma_1$ (resp. $\Sigma_2$) or it is obtained from $\Sigma_1$ (resp. $\Sigma_2$) by repeating some terms. As the series of quotients of $\Sigma_1$, $\Sigma_2$ differ only in the order of the graded pieces, after removing all trivial quotients, the same is true for $\Sigma_1$, $\Sigma_2$.

Example. $G = \mathbb{Z}/6\mathbb{Z}$

$\Sigma_1: \mathbb{Z}/6\mathbb{Z} \triangleright \mathbb{Z}/3\mathbb{Z} \triangleright \mathbb{Z}/2\mathbb{Z} \triangleright \mathbb{Z}/1\mathbb{Z} \triangleright \mathbb{Z}/0\mathbb{Z} \triangleright JH$

$\Sigma_2: \mathbb{Z}/6\mathbb{Z} \triangleright \mathbb{Z}/3\mathbb{Z} \triangleright \mathbb{Z}/2\mathbb{Z} \triangleright \mathbb{Z}/1\mathbb{Z} \triangleright \mathbb{Z}/0\mathbb{Z} \triangleright JH$

graded pieces: $\varphi^{\Sigma_1}_0(G) = \mathbb{Z}/2\mathbb{Z} = \varphi^{\Sigma_2}_0(G)$

$\varphi^{\Sigma_1}_1(G) = \mathbb{Z}/3\mathbb{Z} = \varphi^{\Sigma_2}_1(G)$

Corollary. Let $G$ be a group that admits a JH series. If $\Sigma_1$ is any strictly decreasing composition series of $G$, then there exists a JH series refining $\Sigma_1$.

Sketch of a proof. Let $\Sigma_0$ be a J-H series of $G$. By Schreier's Thom, we can find $\Sigma_0'$ & $\Sigma_1'$ two equivalent composition series refining $\Sigma_0$ & $\Sigma_1$, resp.

The proof of JH Theorem ensures that $\Sigma_0'$ is JH so $\Sigma_1'$ is also JH.

Example 1. $G = \mathbb{Z}/p^k\mathbb{Z}$ $\forall k > 1$ graded pieces for JH $= \mathbb{Z}/p^k\mathbb{Z}$

(simple & order $p^k$)

Example 2. $G = \mathbb{Z}/n\mathbb{Z}$

How to build a JH series for $G$?

- If $n$ is prime, $G$ is simple so $G \triangleright \mathbb{Z}/1\mathbb{Z} \triangleright \mathbb{Z}/0\mathbb{Z}$ is JH.
- If $n$ is not prime, write $n = p_1^{a_1} \cdots p_r^{a_r}$ prime decomposition.

$G \triangleright \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_r^{a_r}\mathbb{Z}$

$\Rightarrow G = \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_r^{a_r}\mathbb{Z}$
\[ G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{ e \} \]

abelian series with graded pieces \( p \)-groups.

We can refine each \( G_i : = \frac{\mathbb{Z}}{n^i} \supseteq G_{i+1} : = \frac{\mathbb{Z}}{p^{i+1}} \) by lifting a JH series \( \frac{\mathbb{Z}}{p^{i+1}} \) (use Example 1).