Lecture 14: Derived & Lower central series, Solvable & Nilpotent groups

Last Time: maximal composition series (Jordan-Hölder)

TODAY: 2 descending series build out of commutators
- → derived series ⤳ solvable gps
- → Lower central series ⤳ nilpotent groups

§1. Derived Series of a group - Solvable groups

Recall: $[G:G] = \langle \underbrace{aba^{-1}b^{-1}}_{=:[a:b]} : a,b\in G\rangle$ commutator subgroup of G.

Definition: given $A, B < G$, we consider
$$[A:B] = \langle aba^{-1}b^{-1} : a\in A, b\in B\rangle$$

Lemma: If $A, B \lhd G$, then $[A,B] \lhd G$.

Proof: For all $g\in G$, $a\in A$, $b\in B$:
$$g\,aba^{-1}b^{-1}g^{-1} = \underbrace{(gag^{-1})}_{\in A}\underbrace{(gbg^{-1})}_{\in B}(gag^{-1})^{-1}(gbg^{-1})^{-1} = \underbrace{[\overset{\in A}{gag^{-1}} : \overset{\in B}{gbg^{-1}}]}_{\in [A,B]} \qquad \square$$

We will use commutators to define a composition series for G in a recursive way:

Corollary: We define recursively:
$$D^0(G) = G \quad , \quad D^{n+1}(G) = D(D^n(G)) := [D^n(G), D^n(G)]$$
Then, each $D^n(G)$ is normal in G, and $D^n(G)/D^{n+1}(G)$ is abelian (by Problem 11 HW1)

Definition: The sequence $\Sigma: G = D^0(G) \geq D^1(G) \geq \cdots\cdots$ is called the derived series of G.

Q: When is Σ a composition series? A: Need $D^N(G) = \{e\}$ for some N.

Definition: We say G is solvable if there exists $N\geq 0$ with $D^N(G)=\{e\}$

Equiv: $\mathcal{D}$: $G \geq D(G) \geq D^2(G) \geq \cdots\cdots \geq D^N(G) = \{e\}$ is a composition series for G & $D^j(G)/D^{j+1}(G)$ is abelian $\forall j$

Remarks: ① The term "solvable" originates from Galois Theory (Math 6112)

② $D^0(G)=\{e\} \iff G$ is trivial.

③ $D^1(G)=\{e\} \iff G$ is abelian (hence, all abelian groups are solvable)

Main examples: abelian grps & p-groups. (next time!)

Example: $G=D_n$, $D_n'=\langle \rho^2\rangle$ which is abelian.

Pf/ • $[s,\rho]=s\rho s^{-1}\rho^{-1}=s\rho s\rho^{-1}=\rho^{-2}$

• $[s\rho^i:\rho^j]=s\rho^i\rho^j(s\rho^i)^{-1}\rho^{-j}=s\rho^{i+j}\rho^{-i}s\rho^{-j}=\rho^{-2j}$

• $[s\rho^i:s\rho^j]=s\rho^i s\rho^j(s\rho^i)^{-1}(s\rho^j)^{-1}=\rho^{j-i}\rho^{-i}s\rho^{-j}s$
$=\rho^{j-2i}\rho^j=\rho^{2(j-i)}$.

• $[\rho^i:\rho^j]=e$

So $\mathcal{D}$: $G=D_n\supseteq D^1(G)=\langle\rho^2\rangle\supseteq D^2(G)=\{e\}$ & $\boxed{D_n \text{ is solvable}}$

Lemma: The group of upper triangular invertible matrices is solvable.

Eg: $B:=\{\begin{pmatrix} a & b\\ 0 & d\end{pmatrix} \quad a,d\in\mathbb{C}^\times, b\in\mathbb{C}\}$
$\cup$
$D^1(B)=\{\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix} : x\in\mathbb{C}\}$
$\cup$
$D^2(B)=\{e\}$

Pf/ • Assume $D^1(B)$ is as claimed, then $D(B)\simeq\mathbb{C}$ abelian
$\Rightarrow D^2(B)=\{e\}$.

• We prove the claim for $D(B)$ by explicit computation

$A=\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}$, $A'=\begin{pmatrix} a' & b'\\ 0 & d'\end{pmatrix}$

$$AA'A^{-1}(A')^{-1}=\frac{1}{ad\,a'd'}\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\begin{pmatrix} a' & b'\\ 0 & d'\end{pmatrix}\begin{pmatrix} d & -b\\ 0 & a\end{pmatrix}\begin{pmatrix} d' & -b'\\ 0 & a'\end{pmatrix}$$

$$= \frac{1}{ada'd'} \begin{pmatrix} aa' & b'a+bd' \\ 0 & dd' \end{pmatrix} \begin{pmatrix} dd' & -b'd-ba' \\ 0 & aa' \end{pmatrix}$$

$$= \frac{1}{ada'd'} \begin{pmatrix} aa'dd' & aa'(b'a+bd'-b'd-ba') \\ 0 & dd'aa' \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad x = \frac{1}{dd'}\left(b'(a-d) - b(a'-d')\right)$$

$\Rightarrow D'(B) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \; x \in \mathbb{C} \right\} \simeq \mathbb{C}$ abelian

Proposition: If G is non-abelian & simple, then $D^n(G) = G$ for all $n \geq 0$, so G is not solvable.

Pf/ $D(G) \neq \{e\}$ (otherwise, G would be abelian), $D(G) \lhd G$ normal $\Longrightarrow D(G) = G$. $\Big\} \Rightarrow D^n(G) = D^0(G) = G$ for all n.

Ex. $D^0(S_n) = A_n$ and A_n is simple nonabelian for $n \geq 5$, so S_n is not solvable for $n \geq 5$.

Pf/ Assume A_n is simple for $n \geq 5$ (next week!). Then

. $[\sigma, \tau] = \sigma\tau\sigma^{-1}\tau^{-1} \in A_n$ for all σ, τ $\Rightarrow D'(S_n) \subseteq A_n$ (HW 4)

. Since S_n is not abelian $D^0(S_n) \neq \{e\}$

. $D'(S_n) \lhd S_n \Longrightarrow \underset{\neq \{e\}}{D'(S_n)} \lhd A_n$ forces $D'(S_n) = A_n$.

$\Rightarrow D^m(S_n) = A_n$ for all $m \geq 1$

Obs: This will be used to show that quintic or higher degree polynomials cannot be solved by radicals (unlike quadratic polynomials in $\mathbb{C}[x]$)

§2. Lower Central Series - Nilpotent groups

We now define a new sequence involving a new commutator.

Set $C^1(G) = G$

$C^{n+1}(G) = [G, C^n(G)] \quad \forall n \geq 1 \quad (\lhd G \text{ if } C^n(G) \lhd G)$

- By induction on n we see $\boxed{C^n(G) \lhd G \quad \forall n}$

Lemma: $C^{n+1}(G) < C^n(G)$ so $C^{n+1}(G) \lhd C^n(G)$

Pf/ $C^{n+1}(G) = \langle \underbrace{gxg^{-1}}_{\substack{\in C^n(G) \\ (C^n(G) \lhd G)}} x^{-1} : g\in G,\ x \in C^n(G) \rangle < C^n(G)$

Since $C^{n+1}(G) \lhd G$, we conclude: $C^{n+1}(G) \lhd C^n(G)$. $\square$

We build the sequence:

$\mathcal{C}: \quad G = C^1(G) \rhd C^2(G) \rhd C^3(G) \rhd \cdots\cdots$ = lower central series

Definition: G is nilpotent if $\exists\ n \geq 1$ such that $C^n(G) = \{e\}$.

Equivalently, $\mathcal{C}$ is a composition series for G.

Examples (1) G abelian, $C^2(G) = (G,G) = 1 \Rightarrow$ nilpotent

(2) $G = D_n$ $\qquad C^2(G) = \langle \rho^2 \rangle = D'(G)$

$C^3(G) = [G, \langle \rho^2 \rangle] = \langle [s\rho^i, \rho^{2j}] : \substack{i=0,\dots,n-1 \\ j=0,\dots,n-1} \rangle = \langle \rho^4 \rangle$

$s\rho^i \rho^{2j} (s\rho^i)^{-1} \rho^{-2j} = s\rho^{2j} s \rho^{-2j} = \rho^{-4j}$

$C^4(G) = [G, \langle \rho^4 \rangle] = \langle \rho^8 \rangle$

By induction: $C^{m+1}(G) = \langle \rho^{2^m} \rangle \quad \forall\, m \geq 1$.

Conclude: D_n is nilpotent if and only if n is a power of 2.

Upshot: Nilpotent groups are solvable. To show this we need some preparation

Remarks: $\mathcal{C}$ satisfies the following properties:

(1) $[G, C^n(G)] = C^{n+1}(G) \quad \forall n$

$C^n(G) \lhd G\ \forall n \quad \& \quad C^{n+1}(G) \lhd C^n(G)\ \forall n$

(2) $C^n(G) / C^{n+1}(G)$ is abelian $\forall n$

(because $[C^n(G) : C^n(G)] \subseteq [G, C^n(G)] = C^{n+1}(G)$. ✓)

(3) $C^2(G) = [G,G] = D'(G)$

(4) $[C^n(G), C^m(G)] \subset C^{n+m}(G)$ (Exercise HW5)

$\Rightarrow$ $\boxed{D^l(G) \subseteq C^{2^l}(G) \quad \text{for all } l \geq 0.}$

Pf/ True for $l=0$ & $l=1$.

$$D^{l+1}(G) = [D^l(G), D^l(G)] \subseteq [C^{2^l}(G), C^{2^l}(G)] \subseteq C^{2^l+2^l}(G) = C^{2^{l+1}}(G)$$

Corollary: Nilpotent $\Rightarrow$ Solvable

⚠ Solvable $\not\Rightarrow$ Nilpotent (eg D_3 is solvable but not nilpotent)

§3 More characterizations of solvable/nilpotent groups

Theorem 1: G is solvable $\Leftrightarrow$ $\exists$ comp series with abelian graded pieces.

Theorem 2: G is nilpotent if and only if it has a composition series

$$\Sigma: \quad G = G_0 \geq G_1 \geq \cdots\cdots \geq G_n = \{e\}$$

with (1) $gr_j^{\Sigma}(G) = G_j/G_{j+1}$ is abelian $\forall j=0,\dots,n-1$

(2) $[G, G_j] \subset G_{j+1}$ $\forall j=0,\dots,n-1$.

Next Time:
- Proofs of these statements
- p-groups are nilpotent
- Sub- & quotients of solvable/nilpotent groups