Lecture 15: Derived & lower central series, Solvable & Nilpotent groups II

Recall: Given $G$ we define a descending series:

1. $D^0(G) > D^1(G) = [G:G] > D^2 = D^1(D(G)) > \ldots$  
   with $D^0(G) = G$ and $D^{j+1}(G) = [D^j(G):D^j(G)] \forall j \geq 0$.

2. $G = C^1(G) > C^2(G) = [G:G] > C^3 = [G:C^2(G)] > \ldots$  
   with $C^1(G) = G$ and $C^{j+1}(G) = [G:C^j(G)] \forall j \geq 0$.

Key Property: $D^j(G) \leq C^{k_j}(G)$ and $C^{j+1}(G)/D^j(G) \cong G$ \forall $j$.

Def.: A group $G$ is solvable if $\mathfrak{D}$ is a comp series ($D^n(G) = \{e\}$ for some $n$).

\text{nilpotent} \quad \mathfrak{D} \quad \text{solvable} \quad (C^n(G) = \{e\} \text{ for some } n)

Proposition: All nilpotent groups are solvable; but $D_3$ is solvable but not nilpotent.

Ex.: $G$ abelian is nilpotent and solvable.

$A_n$ not solvable for $n \geq 5$. (All $n$ is simple & non-abelian for $n \geq 5$)

$D_n$ is solvable but not nilpotent only if $n = 2^k$ for some $k$.

3.1. Testing solvability/nilpotency via composition series:

Theorem 1: $G$ is solvable $\iff$ $\exists$ comp series $\Sigma$ with abelian graded pieces.

Theorem 2: $G$ is nilpotent if and only if it has a composition series

$\Sigma: G = G_0 \geq G_1 \geq \ldots \geq G_n = \{e\}$

with (1) $G_j/G_j \cong \{e\}$ is abelian $\forall j = 0, \ldots, n-1$

(2) $[G_j, G_{j+1}] \leq G_{j+1}$ $\forall j = 0, \ldots, n-1$.

Note: (2) $\Rightarrow$ (1) since $[G_j, G_{j+1}] \leq G_{j+1}$.

Proof (Thm 1). ($\Rightarrow$) Easy since $\mathfrak{D}$ is a comp series of $G$ with abelian graded pieces.
Write the composition series $\Sigma$ as: $G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_n = 1_{\text{trivial}}$

We will show $D^j(G) \subseteq G_j \forall j \in \mathbb{N}$ by induction on $j$. In particular $D^n(G) \subseteq 1_{\text{trivial}}$, so $D^n(G) = 1_{\text{trivial}}$ and $G$ is solvable.

Base case: $j = 0$ is clear since $D^0(G) = G = G_0$.

Inductive step: Assume $D^j(G) \subseteq G_j \forall j < n$. Since $G_j / G_{j+1}$ is abelian, then $D'(G_j / G_{j+1}) = [G_j : G_{j+1}] \subseteq G_{j+1}$, so $D^{j+1}(G) = D'(D^j(G)) \subseteq D'(G_j) \subseteq G_{j+1}$, as we wanted.

$\Box$

Proof of Thm 2: ($\Rightarrow$) Easy since $G$ is a camp series of $G$ with the desired properties by construction + ($\Leftarrow$) ⇒ ($\Rightarrow$).

$\Leftarrow$ It's enough to check that $C^{j+1}(G) \subseteq G_j \forall j = 0, \ldots, n$.

(If so, then $C^{j+1}(G) \subseteq G_n = 1_{\text{trivial}}$)

The claim follows by induction on $j$.

Base case: $j = 0$.

Inductive step: Fix $j > 0$ and assume $C^{j+1}(G) \subseteq G_j \forall j < n$.

$C^{j+2}(G) = [G : C^{j+1}(G)] \subseteq [G : G_j] \subseteq G_{j+1}$ by (2).

Theorem 2. Sub- & quotients:

$Q$: What happens to sub- & quotient objects? Equivalently, to $\text{ses}$?

$\text{Obs}$: If $H < G$, then $D^j(H) \subseteq D^j(G)$ & $C^{j+1}(H) \subseteq C^{j+1}(G) \forall j > 0$.

Proposition 1. Let $G$ be a group & $N < G$. Then, $G$ is solvable if, and only if, $N$ & $G/N$ are.

Equivalent statement. 1 $\longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow \ldots$ $\text{ses}$.

Then $G_2$ is solvable $\iff G_1$ & $G_3$ are solvable.
Example. \( G = D_n, \ N = \langle e \rangle < G \) & \( G/N \cong \mathbb{Z}/2 \mathbb{Z} \) both solvable, so \( D_n \) is solvable.

**Proof:** (\( \Rightarrow \)) First assume \( G \) is solvable & pick \( n > 0 \) with \( D^n(G) = 3G \).
Then \( D^n(N) \leq D^n(G) = 3G \Rightarrow N \) is solvable.

If \( \Pi : G \to G/N \) is the natural projection, then:
\[
\Pi(D(G)) = [G : G] = [\Pi(G) : \Pi(G)] = D\left(G/N\right)
\]
The same argument yields \( \Pi(D^{n'}(G)) = D\left(D^n(G/N)\right) = D^{n'}(G/N) \)
So \( D^n(G/N) = [\Pi(G) : \Pi(G)] = 3G \), thus \( G/N \) is solvable.

(\( \Leftarrow \)) Now, assume \( N \) & \( G/N \) are solvable. By Theorem 1 we have composition series for \( N \) & \( G/N \) with abelian graded pieces
\[
\Sigma: \ N = N_0 \supseteq N_1 \supseteq \ldots \supseteq N_k = \{e\} \quad N_i \text{ abelian } \forall i = 0, \ldots, k-1.
\]
\[
\Sigma': \ G_j = \bar{G}_0 \supseteq \bar{G}_1 \supseteq \ldots \supseteq \bar{G}_s = 3G_{j+1}\}
\]
Set \( \Pi: G \to G/N \) & \( G_j = \Pi^{-1}(\bar{G}_j) \) \( \forall j = 0, \ldots, s \).
So \( G_s = N \), \( G_0 = G \), \( G_{j+1} \triangleleft G_j \) \( \forall j \) & \( G_j/G_{j+1} \sim \bar{G}_j \) abelian.

Set \( G_i = N_i \) \( \forall i = 1, \ldots, k \). Then:
\[
\Sigma'': G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_s = N \supseteq G_{s+1} \supseteq \ldots \supseteq G_{s+k} = 3G \}
\]
is aComposition series for \( G \) with abelian graded pieces. By Thm 1, \( G \) is solvable.

**Proposition 2:**
1. Subgroups & quotients of nilpotent groups are nilpotent.
   [Same proof as for solvable groups.]
2. \( G \) is nilpotent if & only if there is a subgroup \( A \triangleleft Z(G) \) with \( G/A \) nilpotent.
Proof of (2). We only need to show \((\Leftarrow)\). Consider \(\tilde{\pi}: G \rightarrow G/A\) 
(A \not\subseteq G because \(A \subset Z(G)\)) pick \(n\) with \(C^n(G/A) = 1\).

Claim: \(\tilde{\pi}(C^k(G)) = C^k(G/A)\) for \(k = 1, \ldots, n\).

Proof: By induction in \(k\).

- \(k=1\): \(\tilde{\pi}(C^1(G)) = \tilde{\pi}(G) = G/A = C^1(G/A)\)
- Inductive Step: \(\tilde{\pi}(C^{k+1}(G)) = \tilde{\pi}(C^k(G) \cdot C^k(G)) = \tilde{\pi}(C^k(G)) \cdot \tilde{\pi}(C^k(G)) = C^{k+1}(G/A)\).

In particular, \(\tilde{\pi}(C^n(G)) = C^n(G/A) = 1\) \(\Rightarrow\) \(C^n(G) \subseteq A\)

By \(ACZ(G)\) so \(C^{n+1}(G) \subseteq [G:A] = 1\).

⚠️ The last statement fails if \(A\) is not included in \(Z(G)\), i.e., if \(A \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow \ast\) is a sequence, then

- \(G_1, G_3\) nilpotent \(\Rightarrow\) \(G_2\) is nilpotent.

Example: \(G_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, d \in \mathbb{C}, b \in \mathbb{C} \right\}\) 
(see HW5) \(\nabla\) (direct computation)

\[ G_1 = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{C} \right\} \approx \mathbb{C} \]

\[ G_2/G_1 = \mathbb{C} \times \mathbb{C} \] (diagonal entries)

- \(G_1\) & \(G_2/G_1\) are nilpotent (they are abelian!)
- \(G_2\) is solvable but not nilpotent

Corollary: Every \(p\)-group is nilpotent, and thus solvable

Proof: By induction in \(k\) with \(|G| = p^k\).

- \(k=1\): \(G = \mathbb{Z}/p\mathbb{Z}\) is abelian, hence nilpotent
- Inductive Step: \(C^2(G) = [G:G] = \{e\}\)

We know \(Z(G) \neq \{e\}\) so \(Z(G) = p^s\) \(1 \leq s \leq k\).
Case 1: If \( G = Z(G) \), then \( G \) is abelian, hence nilpotent.

Case 2: If \( s < k \), then by inductive hypothesis, \( Z(G) \) is nilpotent and \( \frac{|G|}{|Z(G)|} = p^k \). Thus \( s < k \), so also nilpotent.

By Proposition 2, \( G \) is nilpotent.

**Theorem:** Only finite nilpotent groups are direct products of \( p \)-groups. More precisely, given a finite group \( G \), the following statements are equivalent:

1. \( G \) is nilpotent
2. Every \( p \)-Sylow subgroup of \( G \) is normal
3. \( G \) is a direct product of \( p \)-groups.

**Proof:** \((1) \Rightarrow (2)\) Pick \( H \) a \( p \)-Sylow subgroup of \( G \). If \( H = G \), then \( H \triangleleft G \). Otherwise we have \( H \leq G \).

Claim: \( N_G(H) = G \) (normalizer of \( H \) is the whole group \( G \))

By Lemma 2 below, we have \( H \not\triangleleft N_G(H) \). To prove the claim, we argue by contradiction & assume \( N_G(H) \not\triangleleft G \). Then, the same lemma applied to \( H' = N_G(H) \) gives \( H' \not\triangleleft N_G(H') = N_G(N_G(H)) \).

We show that \( H \not\triangleleft N_G(N_G(H)) \) which will lead to a contradiction, namely \( N_G(N_G(H)) \leq N_G(H) = H' \) & \( H' \not\triangleleft N_G(N_G(H)) \).

To prove \( H \not\triangleleft N_G(N_G(H)) \), we argue by exploiting the fact that \( H \) is a \( p \)-Sylow group of both \( G \) & \( N_G(H) \). (\( |H||N_G(H)| \leq |H||G| \), so \( |G| \text{ } \& \text{ } |N_G(H)| \) involve the same power of \( p \), namely \( |H| \) )

Some observations:

1. \( H \not\triangleleft N_G(H) \) so \( H \) is the unique \( p \)-Sylow subgroup of \( N_G(H) \)
2. If \( h \in H \) & \( g \in N_G(N_G(H)) \), we have \( ghg^{-1} \in N_G(H) \)

Some observations:
But \( \text{ord}(g h g^{-1}) = \text{ord}(h) \) is a power of \( p \) so it's contained in the unique \( p \)-Sylow group \( H \).

Conclude \( g h g^{-1} \in H \) \forall \( g \in N_G(N_G(H)) \) so \( N_G(N_G(H)) \leq N_G(H) \).

Since \( N_G(H) \leq N_G(N_G(H)) \) we get equality so \( H \triangleleft N_G(H) = N_G(N_G(H)) \).

(2) \( \Rightarrow \) (3): We write \( |G| = p_1^{e_1} \cdots p_s^{e_s} \).

Pick \( H_1, \ldots, H_s \) the corresponding unique \( p_i \)-sylow subgroups of \( G \).

We claim \( G \cong H_1 \times \cdots \times H_s \) by induction on \( s \).

First, notice that both groups have the same order since all \( p_i \)'s are pairwise coprime.

Second, notice that \( H_s \triangleleft G \) & by induction on \( s \), we get
\[
H = H_1 \cdots H_{s-1} < G \quad \text{(general version of 3rd Iso Theorem)}
\]
Furthermore, \( H \cong H_1 \times \cdots \times H_{s-1} \) (by induction on \( s \) since each \( H_i \) is normal in \( H \)).

This gives \( |H| = \frac{|G|}{p_s^{e_s}} \) so \( H \cap H_s = \{e\} \) (sizes are coprime!)

To finish, we show \( H \triangleleft G \). This follows by a direct computation since all \( H_1, \ldots, H_{s-1} \triangleleft G \). (For each \( h_i \in H_i \) & \( g \in G \) we have
\[
(g_1 \cdots g_{s-1} g^{-1}) = (g_1 h_1 g')(g_2 h_2 g'') \cdots (g_{s-1} h_{s-1} g') \in H_1 \cdots H_{s-1} = H
\]

Typical element in \( H \).

Then, we have \( H, H_s \triangleleft G \), \( H \cap H_s = \{e\} \) & \( |H H_s| = |G| \), so \( H H_s = G \). This is the definition of direct product.

\( G \cong H \times H_s \cong H_1 \times \cdots \times H_s \) & each \( H_i \) is a \( p_i \)-group.

(3) \( \Rightarrow \) (1) Show that a direct product of nilpotent is nilpotent.
Key steps in this proof:

1. **Lemma 1 (HW5)** If \( N_{1}, N_{2} \triangleleft G \triangleleft (G: N_{1}) \triangleleft N_{2} \triangleleft G \), then we have \( N_{2} \triangleleft H \triangleleft N_{1} \) for all \( H \triangleleft G \).

2. **Lemma 2.** Let \( G \) be a nilpotent group and \( H \triangleleft G \). Then:
   \[
   H \triangleleft N_{G}(H) := \{ g \in G : gHg^{-1} = H \}.
   \]

   **Proof:** Since \( G \) is nilpotent, the lower central series satisfies,
   \[
   G = G_{0} \triangleright G_{1} \triangleright \ldots \triangleright G_{n} = \{e\}
   \]
   with \([G, G_{j}] \triangleleft G_{j+1} \triangleleft G_{j} \triangleleft G \) \( \forall j \).

   Since \([G, G_{j}] \triangleleft G_{j+1} \triangleleft G_{j} \) then \( G_{j+1} \triangleleft G_{j} \) \( \forall j \) by Lemma 1.

   We get \( G = G_{0}H \triangleright G_{1}H \triangleright \ldots \triangleright G_{n}H = H \).

   Fix \( k \) to be the largest index with \( G_{k}H \triangleright G_{k+1}H = H \).

   Then \( H \triangleleft G_{k}H \) and hence \( N_{G_{k}}H \triangleleft G_{k}H \neq H \) as we wanted.

**Corollary:** We understand Jordan-Hölder series of finite nilpotent groups. (Concatenated JH series of each \( p \)-group appropriately.)