# Lecture 15: Derived & Lower central series, Solvable & Nilpotent groups II

**Recall**: Given $G$ we define 2 descending series:

① $\mathcal{D}$: $G = D^0(G) \rhd D^1(G) = [G:G] \rhd D^2 = D^1(D(G)) \rhd \cdots$ derived series

with $D^0(G) = G$ & $D^{j+1}(G) = [D^j(G) : D^j(G)]$ $\forall j \geq 0$.

② $\mathcal{C}$ $G = C^1(G) \rhd C^2(G) = [G:G] \rhd C^3(G) = [G : C^2(G)] \rhd \cdots$ lower central series

with $C^1(G) = G$ & $C^{j+1}(G) = [G : C^j(G)]$ $\forall j \geq 0$.

**Key Property**: $D^j(G) \subseteq C^{2^j}(G)$ & $C^{j+1}(G), D^j(G) \lhd G$ $\forall j$

**Def**: • A group $G$ is solvable if $\mathcal{D}$ is a comp series ( $D^n(G) = \{e\}$ for some $n$)

• ———— nilpotent — $\mathcal{C}$ ———— ( $C^n(G) = \{e\}$ for some $n$ )

**Proposition** All nilpotent groups are solvable ; but $D_3$ is solvable & not nilpotent

**Ex**: • $G$ abelian is nilpotent and solvable

• $S_n$ not solvable for $n \geq 5$. ( $A_n$ is simple & non-abelian for $n \geq 5$)

• $D_n$ is solvable $\forall n$ but nilpotent only if $n = 2^k$ for some $k$

## §1. Testing solvability / nilpotency via composition series:

**Theorem 1**: $G$ is solvable $\iff$ $\exists$ comp series $\Sigma$ with abelian graded pieces.

**Theorem 2**: $G$ is nilpotent if and only if it has a composition series

$$\Sigma : G = G_0 \geq G_1 \geq \cdots \geq G_n = \{e\}$$

with (1) $gr_j^{\Sigma}(G) = G_j / G_{j+1}$ is abelian $\forall j = 0, \ldots, n-1$

(2) $[G, G_j] \subset G_{j+1}$ $\forall j = 0, \ldots, n-1$.

**Note**: (2) $\Rightarrow$ (1) since $[G_j, G_j] \subset G_{j+1}$

**Proof** (Thm 1). ($\Rightarrow$) Easy since $\mathcal{D}$ is a comp series of $G$ with abelian graded pieces.

$(\Leftarrow)$ Write the composition series $\Sigma$ as: $G = G_0 \lhd G_1 \lhd \cdots \cdots \lhd G_n = \{e\}$

We will show $D^j(G) \subseteq G_j$ $\forall j \leq n$ by induction on $j$. In particular $D^n(G) \subseteq \{e\}$ so $D^n(G) = \{e\}$ & $G$ is solvable.

<u>Base case</u>: $j=0$ is clear since $D^0(G) = G = G_0$.

<u>Inductive step</u>: Assume $D^j(G) \subset G_j$ & $j<n$. Since $G_j/G_{j+1}$ is abelian, then $D^1(G_j) = [G_j : G_j] \subseteq G_{j+1}$

So $D^{j+1}(G) = D^1(\underbrace{D^j(G)}) \subseteq \underbrace{D^1(G_j)}_{\substack{\| \\ [G_j : G_j]}} \subseteq G_{j+1}$, as we wanted $\square$

$$D^1(D^j(G)) = [D^j(G) : D^j(G)]$$

<u>Proof of Thm 2</u>: $(\Rightarrow)$ Easy since $\mathscr{C}$ is a comp series of $G$ with the desired properties by construction + $(2) \Rightarrow (1)$.

$(\Leftarrow)$ It's enough to check that $C^{j+1}(G) \subseteq G_j$ $\forall j = 0, \ldots, n$

(If so, then $C^{n+1}(G) \subseteq G_n = \{e\}$)

The claim follows by induction on $j$.

<u>Base case</u>: $j=0$ $\quad C^1(G) = G = G_0$

<u>Inductive step</u>: Fix $j \geq 0$ & assume $C^{j+1}(G) \subseteq G_j$ for $j<n$

$$C^{j+2}(G) = [G : C^{j+1}(G)] \underset{(IH)}{\subseteq} [G : G_j] \subseteq G_{j+1} \text{ by (2).} \quad \square$$

## §2. <u>Sub- & quotients</u>:

Q: What happens to sub- & quotient objects? Equivalently, to ses?

<u>Obs</u>: If $H < G$, then $\underset{\substack{\wedge \\ H}}{D^j(H)} \subseteq \underset{\substack{\wedge \\ G}}{D^j(G)}$ & $\underset{\substack{\wedge \\ H}}{C^{j+1}(H)} \subseteq \underset{\substack{\wedge \\ G}}{C^{j+1}(G)}$ $\forall j \geq 0$ (induct on $j$)

<u>Proposition</u>: Let $G$ be a group & $N \lhd G$. Then, $G$ is solvable if, and only if, $N$ & $G/N$ are.

<u>Equivalent statement</u>: $1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$ ses

Then $G_2$ is solvable $\Longleftrightarrow$ $G_1$ & $G_3$ are solvable.

**Example** $G = D_n$, $N = \langle \rho \rangle \lhd G$ & $G/N \simeq \mathbb{Z}/2\mathbb{Z}$ both solvable (abelian), so $D_n$ is solvable.

**Proof**: ($\Rightarrow$) First assume $G$ is solvable & pick $n \geq 0$ with $D^n(G) = \{e\}$

Then $D^n(N) \subseteq D^n(G) = \{e\} \Rightarrow N$ is solvable.

If $\pi: G \twoheadrightarrow G/N$ is the natural projection, then:

$$\pi(D(G)) = \pi([G:G]) \underset{\pi \text{ gp hom}}{=} [\pi(G):\pi(G)] = D(G/N)$$

The same argument yields $\pi(D^{j+1}(G)) = D(D^j(G/N)) = D^{j+1}(G/N)$

So $D^n(G/N) = \pi(\{e\}) = \{e_{G/N}\}$, thus $G/N$ is solvable

($\Leftarrow$) Now, assume $N$ & $G/N$ are solvable. By Theorem 1 we have composition series for $N$ & $G/N$ with abelian graded pieces

$\Sigma$: $N = N_0 \geq N_1 \geq \cdots \geq N_k = \{e\}$ $\qquad \frac{N_i}{N_{i+1}}$ abelian $\forall i = 0, \ldots, k-1$.

$\Sigma'$: $G/N = \overline{G_0} \geq \overline{G_1} \geq \cdots \geq \overline{G_s} = \{e_{G/N}\}$ $\qquad \overline{G_j}/\overline{G_{j+1}}$ —— $\forall j = 0, \ldots, s-1$.

Set $\pi: G \twoheadrightarrow G/N$ & $\boxed{G_j := \pi^{-1}(\overline{G_j})}$ $\forall j = 0, \ldots, s$ (lifting of $\Sigma'$ to a series for $G$ ending in $N$)

So $G_s = N$, $G_0 = G$, $G_{j+1} \lhd G_j$ $\forall j$ & $G_j/G_{j+1} \simeq \overline{G_j}/\overline{G_{j+1}}$ abelian

2nd iso & $N \lhd G_j$ $\forall j$

Set $\boxed{G_{s+i} := N_i}$ for $i = 1, \ldots, k$. Then:

$\Sigma''$ $G = G_0 \geq G_1 \geq \cdots \geq G_s = N \geq G_{s+1} \geq \cdots \geq G_{s+k} = \{e\}$

is a comp series for $G$ with abelian graded pieces. By Thm 1, $G$ is solvable. $\square$

**Proposition 2:**

① Subgroups and quotients of nilpotent groups are nilpotent

[Same proof as for solvable groups.]

② $G$ is nilpotent if and only if there is a subgroup $A \subset Z(G)$ with $G/A$ nilpotent.

**Proof of ②** We only need to show ($\Leftarrow$). Consider $\pi: G \twoheadrightarrow G/A$ ($A \triangleleft G$ because $A \subset Z(G)$) pick $n$ with $C^n(G/A) = \{e\}$

**Claim:** $\pi(C^k(G)) = C^k(G/A)$ for $k = 1, \dots, n$

**Proof:** By induction on $k$.

- $k=1$: $\pi(C^1(G)) = \pi(G) = G/A = C^1(G/A)$
- Inductive Step: $C^{k+1}(G) = [G : C^k(G)]$ so

$$\pi(C^{k+1}(G)) \underset{\hookrightarrow \pi \text{ surj gp hom.}}{=} [\pi(G) : \pi(C^k(G))] \underset{[IH]}{=} [\tfrac{G}{A} : C^k(G/A)] = C^{k+1}(G/A).$$

In particular, $\pi(C^n(G)) = C^n(G/A) = \{e\} \Rightarrow C^n(G) \subset A$
$\ker \pi = A$

By $A \subset Z(G)$ so $C^{n+1}(G) \subseteq [G : A] = \{e\}$. $\square$

⚠ The last statement fails if $A$ is not included in $Z(G)$,

ie if $1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \rightarrow 1$ ses, then

$G_1, G_3$ nilpotent $\not\Rightarrow$ $G_2$ is nilpotent.

**Example:** (see HW5)

$$G_2 = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, d \in \mathbb{C}^\times \; b \in \mathbb{C} \right\}$$

$\triangledown$ (direct computation)

$$G_1 = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{C} \right\} \simeq \mathbb{C}$$

$$G_2/G_1 = \mathbb{C}^\times \times \mathbb{C}^\times \quad \text{(diagonal entries)}$$

- $G_1$ & $G_2/G_1$ are nilpotent (they are abelian!)
- $G_2$ is solvable but not nilpotent

**Corollary:** Every $p$-group is nilpotent, and thus solvable

**Proof** By induction on $k \geq 1$ with $|G| = p^k$.

- Clear for $k=1$: $G \simeq \mathbb{Z}/p\mathbb{Z}$ so abelian, hence nilpotent ($C^2(G) = [G : G] = \{e\}$)
- Inductive Step:

We know $Z(G) \neq \{e\}$ so $|Z(G)| = p^s$ $1 \leq s \leq k$

CASE 1 If $G = Z(G)$, then $G$ is abelian, hence nilpotent.

CASE 2 If $s < k$, then by inductive hypothesis: $Z(G)$ is nilpotent & $|G/Z(G)| = p^{k-s}$ $k-s<k$, so also nilpotent.

By Proposition 2, $G$ is nilpotent. □

**Theorem**: Only finite nilpotent groups are direct products of p-groups. More precisely, given a finite group $G$, the following statements are equivalent

(1) $G$ is nilpotent

(2) Every p-Sylow subgroup of $G$ is normal

(3) $G$ is a direct product of p-groups.

**Proof**: (1) $\Rightarrow$ (2) Pick $H$ a p-Sylow subgroup of $G$. If $H = G$, then $H \triangleleft G$. Otherwise we have $H \subsetneq G$.

**Claim**: $N_G(H) = G$ (normalizer of $H$ is the whole group $G$)

By Lemma 2 below, we have $H \subsetneq N_G(H)$. To prove the claim, we argue by contradiction & assume $N_G(H) \subsetneq G$. Then, the same lemma applied to $H' = N_G(H)$ gives $H' \subsetneq N_G(H') = N_G(N_G(H))$

• We show that $\boxed{H \triangleleft N_G(N_G(H))}$ which will lead to a contradiction namely $N_G(N_G(H)) \subseteq N_G(H) = H'$ & $H' \subsetneq N_G(N_G(H))$.

To prove $H \triangleleft N_G(N_G(H))$, we argue by exploiting the fact that $H$ is a p-Sylow group of both $G$ & $N_G(H)$. ( $|H| \mid |N_G(H)|$ & $|H| \mid |G|$ so $|G|$ & $|N_G(H)|$ involve the same power of $p$, namely $|H|$ )

Some observations:

① $H \triangleleft N_G(H)$ so $H$ is the unique p-Sylow subgroup of $N_G(H)$

② if $\underset{N_G(H)}{\underset{\cap}{h \in H}}$ & $g \in N_G(N_G(H))$, we have $ghg^{-1} \in N_G(H)$

But $\mathrm{order}(ghg^{-1}) = \mathrm{order}(h)$ is a power of $p$. So it's contained in the unique $p$-Sylow group $H$.

Conclude $ghg^{-1} \in H$ $\forall g \in N_G(N_G(H))$ so $N_G(N_G(H)) \subseteq N_G(H)$.

Since $N_G(H) \subseteq N_G(N_G(H))$ we get equality so

$$H \triangleleft N_G(H) = N_G(N_G(H)).$$

(2) $\Rightarrow$ (3): We write $|G| = p_1^{r_1} \cdots p_s^{r_s}$

Pick $H_1, \ldots, H_s$ the corresponding unique $p_i$-sylow subgroups of $G$.

We claim $G \simeq H_1 \times \cdots \times H_s$ by induction on $s$

- First, notice that both groups have the same order since all $p_i$'s are pairwise coprime.
- Second, notice that $H_s \triangleleft G$ & by induction on $s$, we get $H = H_1 \cdots H_{s-1} < G$ (general version of 3rd Iso Theorem)

  Furthermore, $H \simeq H_1 \times \cdots \times H_{s-1}$ (by induction on $s$ since each $H_i$ is normal in $H$).
- This gives $|H| = \dfrac{|G|}{p_s^{r_s}}$ so $H \cap H_s = \{e\}$ (sizes are coprime!)
- To finish, we show $H \triangleleft G$. This follows by a direct computation, since all $H_1, \ldots, H_{s-1} \triangleleft G$ (for each $h_i \in H_i$ & $g \in G$ we have

$$g \underbrace{h_1 \cdots h_{s-1}}_{\text{typical element in } H} g^{-1} = (gh_1g^{-1})(gh_2g^{-1}) \cdots (gh_{s-1}g^{-1}) \in H_1 \cdots H_{s-1} = H)$$

Then, we have $H, H_s \triangleleft G$, $H \cap H_s = \{e\}$ & $|HH_s| = |G|$, so $HH_s = G$. This is the definition of direct product.

$G \simeq H \times H_s \simeq H_1 \times \cdots \times H_s$ & each $H_i$ is a $p_i$-group.

(3) $\Rightarrow$ (1) Show that a direct product of nilpotents is nilpotent.

Key steps for this proof:

① Lemma 1 (HW5) If $N_1, N_2 \lhd G$ & $(G:N_1) \subset N_2 \subset N_1$, then we have $N_2 H \lhd N_1 H$ for all $H < G$.

② Lemma 2. Let $G$ be a nilpotent group & $H \lneq G$. Then:

$$H \subsetneq N_G(H) := \{g \in G : gHg^{-1} = H\}.$$

Proof: Since $G$ is nilpotent, the lower central series satisfies

$$G = G_0 \supseteq G_1 \supseteq \cdots\cdots \supseteq G_n = \{e\}$$

with $[G, G_j] \subset G_{j+1}$ & $G_j \lhd G$ $\forall j$.

Since $[G, G_j] \subset G_{j+1} \subset G_j$ then $G_{j+1} H \lhd G_j H$ $\forall j$ by Lemma 1

We get $G = G_0 H \underset{\rhd}{\supseteq} G_1 H \underset{\rhd}{\supseteq} \cdots \underset{\rhd}{\supseteq} G_n H = H$.

Fix $k$ to be the largest index with $G_k H \overset{\rhd}{\supsetneq} G_{k+1} H = H$

Then $H \ntrianglelefteq G_k H$ and hence $N_G H \supset G_k H \neq H$ as we wanted.

Corollary: We understand Jordan-Hölder series of finite nilpotent groups. (Concatenated JH series of each $p$-group appropriately)