Lecture 18: Basics on Modules

§1. Modules: Definitions & examples

In what follows, we set R = an arbitrary ring
A = a commutative ring

Def: Left and right modules over R

- A left (resp right) module M (resp. N) over R is an abelian group M (resp. N) together with a bilinear map

$$R \times M \xrightarrow{\cdot} M \qquad (\text{resp } N \times R \xrightarrow{\cdot} N)$$

such that $1 \cdot m = m$ (resp $n \cdot 1 = n$) $\forall a, b \in R$, $m \in M$, $n \in N$

$$(a \cdot b) \cdot m = a \cdot (b \cdot m) \qquad n(a \cdot b) = (n \cdot a) \cdot b$$

Bilinear means linear in each component:

$$(a+b, m) \longmapsto (a+b) \cdot m = (a \cdot m) + (b \cdot m)$$

$$(a, m+m') \longmapsto a \cdot (m+m') = a \cdot m + a \cdot m'.$$

Note: $(-a) \cdot m = -(a \cdot m) = a \cdot (-m)$ from bilinearity

$0_R \cdot m = 0_M$ for all $m \in M$.

Remark: A more economical way of defining left/right modules over R would be to have an abelian group M (resp. N) and a ring hom

$$\lambda : R \longrightarrow \operatorname{End}_{gp}(M) \qquad (\text{resp}: R^{op} \longrightarrow \operatorname{End}_{gp}(N))$$

$$\begin{cases} \lambda(1) = id_M \\ \lambda(r+s) = \lambda(r) + \lambda(s) \\ \lambda(rs) = \lambda(r) \circ \lambda(s) \end{cases}$$

same as R as an abelian gp
$a \cdot b$ in R^{op} $= b\,a$ in R

where $\lambda(r) : M \longrightarrow M$, $m \longmapsto r \cdot m$ (resp $\rho(\lambda) : N \longrightarrow N$, $n \longmapsto n \cdot r$)

Obs: When the ring is commutative, left = right, so we simply use the term module.

Examples: ① $\mathfrak{a} \subset R$ left ideal is a left module / R
right ——— right ———

② Every abelian group is a module over $\mathbb{Z}$

$\left(n\cdot m = \underbrace{m + \cdots + m}_{n \text{ times}} \right.$ for $n \geq 0$ or $\left. n \cdot m = (-n)\cdot(-m) \right)$ for $n \leq 0$.

③ $\forall n \geq 1$: $M = R^n$ (resp $N = R^n$) is a left (resp. right) module over R.

④ R a field ($\mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$), then finite-dimensional vector spaces over R are left R-modules.

§2. Homomorphisms of modules:

Let M_1 & M_2 be two left R-modules. An R-linear map (or left R-module homomorphism) is a homomorphism of abelian groups $f: M_1 \longrightarrow M_2$ such that $f(r\cdot m_1) = r\, f(m_1)$ $\forall\, r \in R,\ m_1 \in M_1$

Write $f \in \operatorname{Hom}_R(M_1, M_2)$ = set of all R-linear maps $M_1 \longrightarrow M_2$.

Obs: $\operatorname{Hom}_R(M_1, M_2)$ has a structure of an abelian gp

$f, g \in \operatorname{Hom}_R(M_1, M_2)$, then $f+g \in \operatorname{Hom}_R(M_1, M_2)$

via $(f+g)(m_1) = f(m_1) + g(m_1) \underset{\uparrow}{=} g(m_1) + f(m_1) =: (g+f)(m_1)$

M_2 ab

. If R is commutative, $\operatorname{Hom}_R(M_1, M_2)$ is an R-module.

- We have the usual notions of submodules, submodules generated by sets, quotient modules, kernels & images. In particular, we have the 3 isomorphism Thms (HW6)

First Iso Thm: $f: M_1 \longrightarrow M_2$ $\rightsquigarrow$
R linear

$$\begin{tikzcd} M_1 \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & M_2 \\ M_1/\ker f \arrow[r, "\cong", "\bar{f}"'] & \operatorname{Im} f \arrow[u, hook] \end{tikzcd}$$

. $\bar{f}$ is an R-linear iso

§3. Direct Sum of modules:

Def Let I be a set and $(M_i)_{i\in I}$ a set of (left) R-modules.

$$\bigoplus_{i\in I} M_i = \{ (x_i)_{i\in I} \;:\; \begin{array}{l} x_i \in M_i \;\; \forall i \\ x_i = 0 \text{ for all but } \underline{\text{finitely many}}\; i \in I \end{array} \}$$

is again a (left) R-module, with componentwise operations:

$$\begin{cases} (x_i)_{i\in I} + (y_i)_{i\in I} = (x_i + y_i)_{i\in I} \\ r\cdot (x_i)_{i\in I} = (r x_i)_{i\in I} \end{cases}$$

Universal Property:

Given a left R-module N and $\{ f_i \in \mathrm{Hom}_R(M_i, N) \}_{i\in I}$, there exists a unique R-linear map

$$f : \bigoplus_{i\in I} M_i \longrightarrow N$$

$$(x_i)_{i\in I} \longmapsto \sum_{i\in I} f_i(x_i)$$ (finite sum by definition of $\bigoplus_{i\in I} M_i$)

$$\begin{tikzcd} \bigoplus_{i\in I} M_i \arrow[rd, dashed, "\exists! f"] & \\ M_j \arrow[u, hook, "\varphi_j"] \arrow[r, "f_j"'] & N \end{tikzcd}$$

Obs: $M_j \xhookrightarrow{\varphi_j} \bigoplus_{i\in I} M_i$ gives $f\circ\varphi_j = f_j$

$m \longmapsto (0,\dots,0,m,0,\dots)$ ← j^{th} spot

Special case: M a left R-module, $M_1, M_2 \subset M$ submodules

Prop: $M \xleftarrow{\sim} M_1 \oplus M_2$ if & only if
- $M_1 + M_2 = M$
- $M_1 \cap M_2 = \{0\}$

Proof: As $M_1 \hookrightarrow M$, $M_2 \hookrightarrow M$ are R-linear, we get by the universal property

$$M_1 \oplus M_2 \xrightarrow{f} M$$

$$(m_1, m_2) \longmapsto m_1 + m_2$$

- Image of f = submodule of M generated by M_1 & M_2
- Kernel of $f = \{ (x, -x) \;:\; x \in M_1 \cap M_2 \}$

Thus, f is an isomorphism iff $M = M_1 + M_2$ & $M_1 \cap M_2 = \{0\}$.

Exercise (HW6): Generalize to $\{M_i \hookrightarrow M\}_{i \in I}$ that is:

$$\bigoplus_{i \in I} M_i \longrightarrow M \quad \text{is an isomorphism iff}$$

(1) $M = \sum_{i \in I} M_i$ (submodule generated by $\{M_i\}_{i \in I}$)

(2) $M_i \cap \sum_{\substack{j \in I \\ j \neq i}} M_j = 0 \quad \forall i \in I$

§4. Direct Product:

Again, if I is a set and $\{M_i\}_{i \in I}$ is a collection of left R-modules, the direct product $\prod_{i \in I} M_i$ is defined as

$$\prod_{i \in I} M_i = \{ (x_i)_{i \in I} \text{ where } x_i \in M_i \ \forall i \}$$

(NOTE: No finiteness condition!)

Remark: For I finite $\bigoplus_{i \in I} M_i \xrightarrow{\sim} \prod_{i \in I} M_i$ as left R-modules. For general I, they are different.

Universal Property: Given a left R-module N and R-linear maps $g_i : N \longrightarrow M_i$, there exists a unique map $N \xrightarrow{g} \prod_{i \in I} M_i$ $\quad g(n) = (g_i(n))_{i \in I}$

not allowed for $\bigoplus_{i \in I}$ if $|I| = \infty$

Furthermore $\pi_j : \prod_{i \in I} M_i \longrightarrow M_j$ is the projection to the j^{th} term, we have

$$\begin{tikzcd} \prod_{i \in I} M_i \arrow[dr, "\pi_j"] & \\ N \arrow[u, "\exists! g"] \arrow[r, "g_j"'] \arrow[ur, phantom, "\circlearrowleft"] & M_j \end{tikzcd} \qquad \forall j \in I$$

§5. Short exact sequences:

Def. If M_1, M_2, M_3 are three left R-modules, and $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ are R-linear maps, we say this sequence is exact (at M_2) if

Image of f = Kernel of g

Def 2: $0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$ s.e.s

means
- f injective , g surjective
- $\mathrm{Im}(f) = \mathrm{Ker}(g)$

Def 3: A short exact sequence $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ is trivial if we have an R-linear isomorphism

$M_1 \oplus M_3 \xrightarrow{\eta} M_2$ st:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "f"] \arrow[d, equal] & M_2 \arrow[r, "g"] & M_3 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_1 \arrow[r, hook, "i"] & M_1 \oplus M_3 \arrow[r, "\pi_2"] \arrow[u, "\eta"] & M_3 \arrow[r] & 0
\end{tikzcd}$$

Proposition: A short exact sequence $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ is trivial if and only if it splits, that is, if $\exists$ R-linear map $s: M_3 \longrightarrow M_2$ st $g \circ s = id_{M_3}$ (an R-linear section)

Proof: ($\Rightarrow$) Take $j: M_3 \hookrightarrow M_1 \oplus M_3$ as the usual inclusion and define $s: M_3 \longrightarrow M_2$ as $\eta \circ j$.

($\Leftarrow$) $\eta: M_1 \oplus M_3 \longrightarrow M_2$ is R-linear

$(x, y) \longmapsto f(x) + s(y)$

and it makes the diagram commute

Exercise: Verify that η is an isomorphism.