Lecture 22: Ideals of ring of fractions, localization

<u>Recall</u>: R commutative, S multiplicatively closed subset of R ($1 \in S$; $0 \notin S$; if $a, b \in S$ then $ab \in S$) gives $S^{-1}R = R \times S/\sim$ with $(a,s) \sim (b,t) \iff \exists r \in S$ with $r(at - bs) = 0$. Write $\frac{a}{s}$ for the class of (a,s)

- $S^{-1}R$ is a ring with $\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$; $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$

  Neutral element: $0 = \frac{0}{1}$ & $1 = \frac{1}{1}$

  <u>Name</u>: Ring of fractions relative to S.

- $j_S : R \longrightarrow S^{-1}R$ $\quad j_S(r) = \frac{r}{1}$ (= class of $(r,1)$) is a ring hom & $j_S(t) = \frac{t}{1}$ is a unit in $S^{-1}R$ for each $t \in S$, with inverse $\frac{1}{t}$.
- Same construction for module of fractions $\underset{R\text{-module}}{M} \rightsquigarrow \underset{S^{-1}R\text{-module}}{S^{-1}M} = M \times S/\sim$

§1. Ideals in $S^{-1}R$:

Fix R commutative ring, $S \subset R$ mult. closed & $j_S : R \to S^{-1}R$ the natural ring homomorphism.

Given $\mathfrak{a} \subset R$ ideal, we define:

$$\boxed{S^{-1}\mathfrak{a}} = \text{ideal in } S^{-1}R \text{ generated by } j_S(\mathfrak{a}).$$

<u>Theorem</u>: Every ideal in $S^{-1}R$ is of this form ($= S^{-1}\mathfrak{a}$ for some $\mathfrak{a} \subset R$ ideal)

Furthermore, $S^{-1}\mathfrak{a} = S^{-1}R \iff S \cap \mathfrak{a} \neq \emptyset$.

<u>Proof</u>: Use $j_S : R \longrightarrow S^{-1}R$

Let $\mathfrak{b} \subset S^{-1}R$ be an ideal & set $\mathfrak{a} = j_S^{-1}(\mathfrak{b})$.

- We know $\mathfrak{a}$ is an ideal because j_S is a ring homomorphism
- <u>Claim</u>: $S^{-1}\mathfrak{a} = \mathfrak{b}$

  Pf/ ($\subseteq$) $a \in \mathfrak{a}$, $\frac{r}{s} \in S^{-1}R \Rightarrow \frac{r}{s} \cdot \frac{a}{1} = \underbrace{\boxed{\frac{r}{s}}}_{\in S^{-1}R} \overbrace{\boxed{\frac{a}{1}}}^{\in \mathfrak{b}} \in \mathfrak{b}$ (ideal)

  So $S^{-1}\mathfrak{a} = S^{-1}R\left(\frac{a}{1} : a \in \mathfrak{a}\right) \subseteq \mathfrak{b}$.

($\supseteq$) Pick $x \in \mathfrak{b} \subset S^{-1}R$. Then, $x = \frac{y}{s}$ for some $y \in R$, $s \in S$

$\Rightarrow \frac{y}{1} = y = \underbrace{\boxed{\frac{s}{1}}}_{\in R} \cdot \underbrace{\boxed{\frac{y}{s}}}_{\in \mathfrak{b}} \in \mathfrak{b}$ so $y \in \mathfrak{a}$ so $\mathfrak{b} \subseteq S^{-1}\mathfrak{a}$.

For the last part:

If $s \in S \cap \mathfrak{a}$, write $1 = \frac{s}{s} \in S^{-1}\mathfrak{a}$. ($\frac{1}{1} = \frac{s}{s}$ because $1(s \cdot 1 - 1 \cdot s) = 0$)

Conversely, assume $1 \in S^{-1}\mathfrak{a}$. Then, $\exists$ $a_1, \ldots, a_n \in \mathfrak{a}$ & $\frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n} \in S^{-1}R$ with $1 = \sum_{i=1}^{n} \frac{r_i}{s_i} \cdot a_i = \sum_{i=1}^{n} \frac{(r_i a_i)}{s_i} = \sum_{i=1}^{n} \frac{b_i}{s}$

where $s = \prod_{i=1}^{n} s_i$ & $b_i = r_i a_i \prod_{j \neq i} s_j \in \mathfrak{a}$ $\forall i = 1, \ldots, n$

BUT $\sum_{i=1}^{n} \frac{b_i}{s} = \frac{\sum_{i=1}^{n} b_i}{s} = \frac{b}{s}$ with $b = \sum_{i=1}^{n} b_i \in \mathfrak{a}$

So $1 = \frac{b}{s}$ for $b \in \mathfrak{a}$, $s \in S$ $\Longleftrightarrow$ $\exists t \in S$ with

$0 = t(s \cdot 1 - b \cdot 1) = ts - tb$

So $\underset{S}{\underset{\cap}{ts}} = \underset{\mathfrak{a}}{\underset{\cap}{tb}} \in S \cap \mathfrak{a}$. $\square$

Proposition: Prime ideals in $S^{-1}R$ are of the form $S^{-1}\mathfrak{p}$, where $\mathfrak{p} \subsetneq R$ is a prime ideal with $\mathfrak{p} \cap S = \emptyset$.

Proof Let $\mathfrak{q} \subsetneq S^{-1}R$ be a prime ideal. By the proof of the previous Theorem, we know $\mathfrak{q} = S^{-1}\mathfrak{a}$ where $\mathfrak{a} = j_S^{-1}(\mathfrak{q})$.

Since $\mathfrak{q}$ is prime & j_S is a ring homomorphism, we know $\mathfrak{a}$ is a prime ideal of R. Since $\mathfrak{q} \subsetneq S^{-1}R$, we must have $\mathfrak{a} \cap S = \emptyset$

Conversely, given $\mathfrak{p} \subsetneq R$ prime with $\mathfrak{p} \cap S = \emptyset$, we want to show $S^{-1}\mathfrak{p} \subseteq S^{-1}R$ is a prime ideal.

. Properness follows since $\mathfrak{P} \cap S = \emptyset$

. $S^{-1}\mathfrak{P}$ is an ideal of $S^{-1}R$ by the Theorem.

. $\frac{a}{s} \cdot \frac{b}{t} \in S^{-1}\mathfrak{P}$ with $a, b \in R$ $s, t \in R$ we get

$\frac{ab}{1} = \left(\frac{st}{1}\right) \frac{ab}{st} \in S^{-1}\mathfrak{P} \Rightarrow ab \in j_S^{-1}(S^{-1}\mathfrak{P}) = \mathfrak{P} \overset{\mathfrak{P} \text{ prime}}{\Longrightarrow} a \in \mathfrak{P}$ or $b \in \mathfrak{P}$.

$\Rightarrow \frac{a}{s} \in S^{-1}\mathfrak{P}$ or $\frac{b}{t} \in S^{-1}\mathfrak{P}$.

Summary: Rings & modules of fractions for R commutative ring

$S \subset R$ mult. closed set $\rightsquigarrow$ $S^{-1}R$ = another comm ring

(HW7) M : R-module $\rightsquigarrow$ $S^{-1}M$: an $S^{-1}R$-module.

Prop: (1) Every ideal of $S^{-1}R$ is of the form $S^{-1}\mathfrak{a}$ for $\mathfrak{a} \subset R$ ideal & $S^{-1}\mathfrak{a} = S^{-1}R \iff S \cap \mathfrak{a} \neq \emptyset$.

(2) Prime ideals of $S^{-1}R \xleftrightarrow{1-1}$ prime ideals of A not meeting S

$S^{-1}R(j_S(\mathfrak{P})) = S^{-1}\mathfrak{P} \longleftarrow \mathfrak{P}$

§2. Localization:

GOAL: Build suitable S for which $S^{-1}R$ is a local ring (unique max'l ideal)

Geometrically: study a space X in the neighborhood of a point.
$R = \{$polynomial functions $X \longrightarrow \mathbb{C}\}$.

Fix $\mathfrak{P} \subsetneq R$ a prime ideal and let $S = R \setminus \mathfrak{P}$.

Lemma: S is multiplicatively closed.

Pf/ . $1 \in S$ $(1 \notin \mathfrak{P})$ & . $0 \notin S$ $(0 \in \mathfrak{P})$

. $a, b \in S$ means $a, b \notin \mathfrak{P}$ So $ab \notin \mathfrak{P}$ because $\mathfrak{P}$ is prime $\Rightarrow ab \in S$. ∎

Def: $R_{\mathfrak{P}} := S^{-1}R$ is called the localization of R at the prime ideal $\mathfrak{P}$.

Proposition: $R_\mathfrak{p}$ is a local ring with unique maximal ideal $\mathfrak{p}R_\mathfrak{p}$.

Proof: Let $\mathfrak{b}$ be a proper ideal of $R_\mathfrak{p}$. By Prop (1) $\mathfrak{b} = S^{-1}\mathfrak{a}$ for $\mathfrak{a} \subset R$ ideal. If $\mathfrak{a} \cap S \neq \emptyset$, then $\mathfrak{b} = S^{-1}R$. Contr!
So $\mathfrak{a} \cap S = \emptyset$, meaning $\mathfrak{a} \subset \mathfrak{p}$. Hence $\mathfrak{b} \subseteq \mathfrak{p}(S^{-1}R)$
So every proper ideal of $R_\mathfrak{p}$ lies in $\mathfrak{p}R_\mathfrak{p}$. Thus $(R_\mathfrak{p}, \mathfrak{p}_\mathfrak{p})$ is local $\square$

Obs: If R is a domain, $j_S : R \hookrightarrow \mathrm{Quot}(R) = (R \setminus \{0\})^{-1}R$
So $R \hookrightarrow R_\mathfrak{p} \hookrightarrow \mathrm{Quot}(R)$

Examples ① $R = \mathbb{Z}$ $\quad (p)$ is prime ideal $\rightsquigarrow S = \{b \in \mathbb{Z} : p \nmid b\}$

$\rightsquigarrow \mathbb{Z}_{(p)} = \{\frac{a}{b} : a, b \in \mathbb{Z} \ \gcd(a,b) = 1, \ p \nmid b\}$ (HW 7)

② $R = \mathbb{C}[x]$ $\quad (x)$ is a maxl ideal, so it's prime

$$R_{(x)} = \left\{ \frac{P}{Q} : P, Q \in \mathbb{C}[x] : x \nmid Q \right\}$$

③ $R = \mathbb{C}[x,y]$ $\quad \mathfrak{m} = (x,y)$ is maxl ideal, so prime

$$R_{(x,y)} = \left\{ \frac{P}{Q} : P, Q \in \mathbb{C}[x,y] \quad Q(0,0) \neq 0 \right\}$$

Def: Given M an R-module, we define its localization at $\mathfrak{p}$ as $M_\mathfrak{p} = S^{-1}M$ where $S = R \setminus \mathfrak{p}$.

Q: What is $\ker\left(M \xrightarrow{i_S} S^{-1}M \right)$? $\quad m \longmapsto \frac{m}{1}$

A: $\ker(i_S) = \{ m \in M : \exists s \in S \text{ with } s \cdot m = 0 \text{ in } M \}$

Def: $\mathrm{Ann}(m) = \{ r \in R : rm = 0 \}$ (Annihilator of m)

Lemma: $\mathrm{Ann}(m)$ is an ideal of R. It is proper $\Leftrightarrow m \neq 0$. (because $1 \cdot m = m$)

Consequence: $m \in \ker(i_S) \Leftrightarrow \mathrm{Ann}(m) \cap S \neq \emptyset$.

Localizations are useful tools to decide when modules are Trivial. More precisely:

Theorem 1: (1) $M = \{0\} \iff$ (2) $M_{\mathfrak{p}} = 0 \quad \forall\, \mathfrak{p} \subsetneq R$ prime ideal
$\iff$ (3) $M_{\mathfrak{m}} = 0 \quad \forall\, \mathfrak{m} \subsetneq R$ mxl ideal

Proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear (mxl ideals are prime)
To finish, we prove (3) $\Rightarrow$ (1): We argue by contradiction.
Pick $m \in M \setminus \{0\}$ & let $\mathfrak{a} = \mathrm{Ann}(m) \subsetneq R$. Pick $\mathfrak{m} \subset R$ maximal ideal with $\mathfrak{a} \subset \mathfrak{m}$. By hypotheses $M_{\mathfrak{m}} = 0$, so $\frac{m}{1} = 0$ in $M_{\mathfrak{m}}$ meaning $\exists\, s \in R \setminus \mathfrak{m}$ with $sm = 0$. This cannot happen since $(R \setminus \mathfrak{m}) \cap \mathrm{Ann}(m) = \emptyset$. $\square$

Theorem 2: Assume R is an integral domain. Then:

$$R = \bigcap_{\mathfrak{m} \text{ mxl ideal}} R_{\mathfrak{m}} = \bigcap_{\mathfrak{p} \text{ prime ideal}} R_{\mathfrak{p}}$$

Proof: Next Time.