Lecture 23: Modules of fractions; Noetherian rings

§1 Localization:

Recall: R commutative ring, $\mathfrak{p} \subset R$ prime ideal

Then: $S := R \setminus \mathfrak{p} \subset R$ is a mult. closed set

$R_{\mathfrak{p}} = S^{-1}R$ is a local ring with mxl ideal $R_{\mathfrak{p}}\mathfrak{p} = S^{-1}\mathfrak{p}$.

Key observation: If R is a domain, then:

$$R \hookrightarrow R_{\mathfrak{p}} \hookrightarrow \operatorname{Quot}(R) =: (R \setminus \{0\})^{-1} R$$
$$r \longmapsto \frac{r}{1} \qquad \frac{a}{b} \longmapsto \frac{a}{b}$$

We view R & $R_{\mathfrak{p}}$ as subrings of $\operatorname{Quot}(R)$

Theorem 1: Assume R is an integral domain. Then:

$$R = \underset{\substack{\mathfrak{p} \text{ prime} \\ \text{ideal}}}{\bigcap} R_{\mathfrak{p}} = \underset{\substack{\mathfrak{m} \text{ mxl} \\ \text{ideal}}}{\bigcap} R_{\mathfrak{m}}$$

$\subseteq \qquad \subseteq$ viewed in Quot$R$ by key Observation

Proof: Write $\tilde{R} = \bigcap_{\mathfrak{m} \text{ mxl}} R_{\mathfrak{m}} \supseteq R$. We view $\tilde{R}/R$ as an R-module

Then: $\tilde{R} = R \iff \tilde{R}/R = 0$ as an R-module

Since $\tilde{R} \subset \operatorname{Quot}(R)$ we write $\bar{r} \in \tilde{R}/R \subset \frac{\operatorname{Quot}(R)}{R}$ as $\overline{\frac{a}{b}}$ with $\frac{a}{b} \in \operatorname{Quot}(R)$

- We want to show $a \in (b)$, so $\frac{a}{b} \in R$.

Consider $I = \{t \in R : t\frac{a}{b} \in R\} = \operatorname{Ann}(\overline{\frac{a}{b}})$

This means $\frac{ta}{b} = \frac{a'}{1}$ for $a' \in R$, i.e. $ta = a'b$

Thus $I = \{t \in R : ta \in R(b)\} = (R(b) : a)$ since R is a domain

- If $I = (1)$, then $a = 1 \cdot a \in (b)$
- Otherwise, $\exists\ \mathfrak{m}$ mxl ideal of R with $I \subseteq \mathfrak{m} \subsetneq R$. Since $r - \frac{a}{b} \in R \subseteq \tilde{R}$ ($\in \tilde{R}$)

we have $\frac{a}{b} \in \tilde{R} \subseteq R_{\mathfrak{m}}$ So $\frac{a}{b} = \frac{a'}{b'}$ with $b' \notin \mathfrak{m}$

$b'a = a'b$ so $b' \in I \subseteq \mathfrak{m}$ Contradiction! ($b' \in I \subset \mathfrak{m}$ & $b' \notin \mathfrak{m}$) $\square$

§2. Modules of fractions and their homomorphisms:

Recall: Fix R commutative ring, $S \subset R$ mult closed set, M an R-module

$\leadsto$ $S^{-1}M$ = module of fractions relative to S $= M \times S/\sim$ $\quad (m,s) \sim (m',s')$
(module over $S^{-1}R$) $\quad \Leftrightarrow \exists r \in S$ with $r(s'm - s \cdot m') = 0 \in M$

Write $\frac{m}{s}$ for the class of (m,s).

- Let M, N be two R-modules & set $f: M \longrightarrow N$ R-linear

Then $S^{-1}f: S^{-1}M \longrightarrow S^{-1}N$ is $S^{-1}R$ linear ("extend scalars from R to $S^{-1}R$")
$\frac{m}{s} \longmapsto \frac{f(m)}{s}$

Proposition: Let $0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$ be a ses of R-linear maps between R-modules. Then, the following sequence of $S^{-1}R$-linear maps is exact:

$$0 \longrightarrow S^{-1}M_1 \xrightarrow{S^{-1}f} S^{-1}M_2 \xrightarrow{S^{-1}g} S^{-1}M_3 \longrightarrow 0.$$

("localization is an exact functor")

Proof: (1) $\operatorname{Ker}(S^{-1}f) = \{\frac{m}{s} \in S^{-1}M_1 : \frac{f(m)}{s} = 0 \text{ in } S^{-1}M_2\}$

Pick $\frac{m}{s} \in \operatorname{ker}(S^{-1}f)$, so $\frac{f(m)}{s} = 0$. Then, $\frac{f(m)}{1} = s \cdot \frac{f(m)}{s} = s \cdot \frac{0}{1} = \frac{s \cdot 0}{1} = \frac{0}{1}$

$\Rightarrow \exists r \in S$ with $r(1 \cdot f(m) - 1 \cdot 0) = rf(m) = 0$ so $f(rm) = rf(m) = 0$.

$\Rightarrow rm \in \operatorname{Ker}(f) = 0$. Then $\frac{m}{s} = \frac{rm}{rs} = \frac{0}{rs} = \frac{0}{1}$

Conclude: $\operatorname{Ker}(S^{-1}f) = \{0\}$.

(2) $S^{-1}g$ is surjective: Let $\frac{m_3}{s} \in S^{-1}M_3$, $m_3 \in M_3$, $s \in S$

Since g is surjective $\exists m_2 \in M_2$ st $g(m_2) = m_3$

So $\frac{m_3}{s} = \frac{g(m_2)}{s} = S^{-1}g\left(\frac{m_2}{s}\right)$. Conclude: $S^{-1}g$ is surjective.

(3) $\operatorname{Ker}(S^{-1}g) = \operatorname{Im}(S^{-1}f)$.

($\supseteq$) $(S^{-1}g) \circ (S^{-1}f)\left(\frac{m_1}{s}\right) = S^{-1}(g)\left(\frac{f(m_1)}{s}\right) = \frac{g(f(m_1))}{s} = \frac{0}{s} = 0$

So $\operatorname{Im}(S^{-1}f) \subseteq \operatorname{Ker}(S^{-1}g)$

($\subseteq$) Conversely, if $\frac{m_2}{s} \in \operatorname{Ker}(S^{-1}g)$ then $\frac{g(m_2)}{s} = \frac{0}{1}$ so

$\delta(m_2) = s\,\frac{g(m_2)}{s} = \frac{0}{1}$. By def, $\exists\, r \in S$ with $r g(m_2) = 0$ in M_3

So $r m_2 \in \ker g = \operatorname{Im} f$, ie $r m_2 = f(m_1)$ for some $m_1 \in M_1$

Then $\frac{m_2}{s} = \frac{r m_2}{rs} = \frac{f(m_1)}{rs} = S^{-1}f\left(\frac{m_1}{rs}\right) \in \operatorname{Im} S^{-1}f$. $\square$

<u>Obs</u>: We can use this to give an alternative proof of Thm 1 (see HW8)

<u>Corollary</u>: (1) Let $N \subset M$ be submodule over R.

Then $\frac{S^{-1}M}{S^{-1}N} \simeq S^{-1}(M/N)$. (as $S^{-1}R$-modules)

(2) In particular, for an ideal $\mathfrak{a} \subset R$, we have $S^{-1}R / S^{-1}\mathfrak{a} \simeq S^{-1}(R/\mathfrak{a})$ as $S^{-1}R$-modules.

(3) If $S \cap \mathfrak{a} = \emptyset$, then $\overline{S}$ = image of S under $R \longrightarrow R/\mathfrak{a}$ is multiplicatively closed & $\overline{S}^{-1}(R/\mathfrak{a})$ is a ring & an $S^{-1}R$-module.

Moreover, $S^{-1}(R/\mathfrak{a}) \xrightarrow{\Psi} \overline{S}^{-1}(R/\mathfrak{a})$, $\Psi\left(\frac{\overline{r}}{s}\right) = \frac{\overline{r}}{\overline{s}}$ is an iso of $S^{-1}R$-modules

Pf/ (1) Use $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ ses of R-mod

Then $0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N) \longrightarrow 0$ is ses of $S^{-1}R$-mod

(2) Is a special case of (1): ideals of R are R-modules.

(3) $\overline{S}$ is mult. closed in $R/\mathfrak{a}$ • $\overline{1} \in \overline{S}$ ✓ • $0 \notin \overline{S} \iff S \cap \mathfrak{a} = \emptyset$ ✓ • $\overline{a}, \overline{b} \in \overline{S} \Rightarrow \overline{a}\,\overline{b} = \overline{ab} \in \overline{S}$ ✓

Then: $\overline{S}^{-1}(R/\mathfrak{a})$ is the ring of fractions of $R/\mathfrak{a}$ relative to $\overline{S}$.

But it is also an $S^{-1}R$-module via $\underset{S^{-1}R \ni}{\frac{a}{b}} \cdot \underset{\in \overline{S}^{-1}(R/\mathfrak{a})}{\frac{\overline{r}}{\overline{s}}} := \frac{\overline{ar}}{\overline{bs}} \in \overline{S}^{-1}(R/\mathfrak{a})$

• <u>Ψ is well-def</u>: $\frac{\overline{r}}{s} = \frac{\overline{r'}}{s'} \iff \exists\, t \in S$ st $t(s'\overline{r} - s\overline{r'}) = 0$ in $R/\mathfrak{a}$.

$\iff t(s'r - sr') \in \mathfrak{a}$ & $t \in S \implies \overline{t}(\overline{s'}\,\overline{r} - \overline{s}\,\overline{r'}) = \overline{0}$ in $R/\mathfrak{a}$ & $\overline{t} \in \overline{S}$.

$\implies \frac{\overline{r}}{\overline{s}} = \frac{\overline{r'}}{\overline{s'}}$ in $\overline{S}^{-1}(R/\mathfrak{a})$.

• The map Ψ is $S^{-1}R$-linear by construction.

• <u>Ψ is injective</u>: $\frac{\overline{r}}{\overline{s}} = 0 \iff \exists\, \overline{s'} \in \overline{S}$ with $\overline{s'}(\overline{r}\cdot\overline{1}) = 0$ in $R/\mathfrak{a}$

$\iff \exists\, s' \in S$ — $s'(r\cdot 1) \in \mathfrak{a}$

$\iff \overline{r} = 0$ in $S^{-1}(R/\mathfrak{a}) \implies \frac{\overline{r}}{s} = 0$ in $S^{-1}(R/\mathfrak{a})$

• <u>Ψ is surjective</u> by construction: $\frac{\overline{r}}{\overline{s}} = \Psi\left(\frac{\overline{r}}{s}\right)$. $\square$

§3. Finiteness properties of rings: Noetherian & Artinian rings

In 1888, Kronecker published his findings on "ideal = product of prime ideals" research. He made a crucial assumption for ideals over polynomial rings: that they are finitely generated. This fact was only later proven by Hilbert (Hilbert Basis Thm). This property of rings ("every ideal is finitely generated") has the following axiomatization

<u>Definition</u>: A commutative ring R is called <u>Noetherian</u> if for every chain of ideals of R $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \cdots$ there is $k \geq 0$ with $\mathfrak{a}_k = \mathfrak{a}_{k+1} = \cdots$ [ACC = Ascending chain condition]

<u>Theorem 1</u>: The following conditions on a commutative ring R are equivalent:

(1) R is Noetherian

(2) Every nonempty set $\mathcal{S}$ of ideals of R has a maximal element

(3) Every ideal $\mathfrak{a} \subseteq R$ is finitely generated.

<u>Proof</u>: $\boxed{(1) \Rightarrow (2)}$ Let $\mathfrak{a}_0 \in \mathcal{S}$. If $\mathfrak{a}_0$ is not maximal, $\exists$ $\mathfrak{a}_1 \in \mathcal{S}$ with $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1$. Continuing in this fashion, we get an ascending chain of ideals $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \cdots$ that doesn't stabilize. <u>Contr!</u> Then $\exists$ $\mathfrak{a}_k$ maximal element of $\mathcal{S}$.

$\boxed{(2) \Rightarrow (3)}$ Let $\mathfrak{a}$ be an ideal. Consider the set

$$\mathcal{S} = \{ \mathfrak{a}' \subseteq \mathfrak{a} : \mathfrak{a}' \text{ is a fin. gen. ideal of } R \}$$

We order $\mathcal{S}$ by inclusion.

By (2), this set has a maximal element, say $\tilde{\mathfrak{a}}$.

If $\tilde{\mathfrak{a}} \subsetneq \mathfrak{a}$, pick $x \in \mathfrak{a} \setminus \tilde{\mathfrak{a}}$. Then $(\tilde{\mathfrak{a}}, x) \in \mathcal{J}$, contradicting the maximality of $\tilde{\mathfrak{a}} < (\tilde{\mathfrak{a}}, x)$. Hence $\tilde{\mathfrak{a}} = \mathfrak{a}$, which means $\mathfrak{a}$ is finitely generated since $\tilde{\mathfrak{a}} \in \mathcal{J}$.

$\boxed{(3) \Rightarrow (1)}$ Let $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \dots$ be a chain of ideals of R.

Take $\mathfrak{a} = \bigcup_{i=0}^{\infty} \mathfrak{a}_i \subset R$.

By construction, $\mathfrak{a}$ is an ideal of R, thus finitely generated by elements $a_1, \dots, a_n \in \mathfrak{a}$. Now, each $a_\ell \in \mathfrak{a}_{j_\ell}$ for some $j_\ell \geq 0$. Thus, $\mathfrak{a} = \mathfrak{a}_j$ for $j = \max\{j_1, j_2, \dots, j_n\}$ and so $\mathfrak{a} = \mathfrak{a}_j = \mathfrak{a}_{j+1} = \dots$ The chain terminates, so A is Noetherian.