§ 1. Noetherian Rings:

Recall: A commutative ring $R$ is Noetherian if it satisfies ACC (every ascending chain of ideals stabilizes).

**Theorem:** If $R$ is a commutative ring, the following are equivalent:

1. $R$ is Noetherian.
2. Every non-empty set $S$ of ideals of $R$ has a maximal element (with respect to inclusion).
3. Every ideal $a \subseteq R$ is finitely generated.

**Corollary.** (1) Principal ideal domains are Noetherian (e.g. $\mathbb{Z}, \mathbb{C}[x]$)

2. If $f: A \to B$ is a ring homomorphism with $A$, $B$ commutative, $A$ is Noetherian, so is $B$.

3. Rings of fractions of Noetherian rings are Noetherian. In particular, localizations preserve Noetherianness.

**Proof.** (2): If $I \subseteq B$ is an ideal, $\alpha = f^{-1}(I) \subseteq A$ is an ideal, so $\alpha = (a_1, \ldots, a_n)$. Then $I = f(\alpha) = (f(a_1), \ldots, f(a_n))$.

**Warning:** Subrings of Noetherian rings need not be Noetherian.

**Ex:** Take $R = \mathbb{C}[x_1, x_2, \ldots] = \bigcup_{n \in \mathbb{N}} \mathbb{C}[x_1, x_2, \ldots, x_n]$

Note: $R$ is not Noetherian since the ascending chain of ideals

$\alpha_1 = (x_1) \subseteq \alpha_2 = (x_1, x_2) \subseteq \ldots$ never terminates.

- Now: $R$ is a domain & $R \to Quot(R) = \text{field}$, so $R$ is a subring of $Quot(R)$.
- Since the only ideals of $Quot(R)$ are $(0)$ & $(1)$, we get that $Quot(R)$ is Noetherian.
Hilbert Basis Thm: If $R$ is Noetherian, so is $R[x]$.

Hence $\mathbb{Z}[x_1, \ldots, x_n] \cong K[x_1, \ldots, x_n]$ is Noetherian in any field $K$.

To prove this result, we'll need the notion of Noetherian modules.

§ 2. Noetherian modules:

Fix $R$ a commutative ring, and $M$ an $R$-module.

Def: We say $M$ is Noetherian if it satisfies the ascending chain condition for submodules:

"Every chain of submodules $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ stabilizes 
 i.e. $\exists \ell \geq 0$ s.t. $M_\ell = M_{\ell+1} = \cdots$"

We have the following analog of Theorem 1:

Theorem 2: Fix $R$ a commutative ring, and $M$ an $R$-module. TFAE:

1. $M$ is Noetherian
2. Every nonempty set $\mathfrak{L}$ of submodules of $M$ has a maximal element (with respect to inclusion)
3. Every submodule of $M$ is finitely generated.

The proof is exactly the same as that of Thm 1.

Corollary 2: Let $0 \longrightarrow M_1 \overset{f}{\longrightarrow} M_2 \overset{g}{\longrightarrow} M_3 \longrightarrow 0$ be a SES of $R$-modules. Then: $M_2$ is Noetherian if and only if $M_1$ and $M_3$ are.

Proof: $\Rightarrow$) submodules of $M_1$ are submodules of $M_2$ via $f$.

$N_3 \subseteq M_3$ come from $g^{-1}(N_3) \subseteq M_2$ submodules

$g^{-1}(N_3) = \langle m_1, \ldots, m_2 \rangle$, then $N_3 = \langle g(m_1), \ldots, g(m_2) \rangle$ by surjectivity of $g$.

$\Leftarrow$) Pick $N$ a submodule of $M_2$, then $g(N) \subseteq M_3$ is a submodule
So \( g(N) = \langle m, \ldots, m\rangle \). Pick \( n_1, \ldots, n_5 \in N \) with 
\[
g(n_1) = m_1, \ldots, g(n_5) = m_5.
\]

Next take \( f(N \cap f(M_1)) = N_1 \leq M_1 \) submodule, so it's finitely generated 
\( N_1 = \langle n_1, \ldots, n_e \rangle \). Then \( n'_1 = f(n_1), \ldots, n'_e = f(n_e) \). & \( N \cap f(M_1) = \langle n'_1, \ldots, n'_e \rangle \)

Claim: \( N = \langle n_1, \ldots, n_5, n'_1, \ldots, n'_e \rangle \)

\[ \text{Proof:} \] Pick \( n \in N \), so \( g(n) \in g(N) = \langle m, \ldots, m\rangle \)
That is \( g(n) = q_1 m_1 + \cdots + q_5 m_5 = q_1 g(n_1) + \cdots + q_5 g(n_5) \) \( = g(q_1 n_1 + \cdots + q_5 n_5) \quad q_1, \ldots, q_5 \in R \)
This means \( n - q_1 n_1 - \cdots - q_5 n_5 \in \text{Ker } g = \text{Im } f \), so 
\( n - q_1 n_1 - \cdots - q_5 n_5 \in N \cap f(M_1) = \langle n'_1, \ldots, n'_e \rangle \)

Conclude \( n \in \langle n_1, \ldots, n_5, n'_1, \ldots, n'_e \rangle \)

\[ \square \]

Note:
1. \( R \) is an \( R \)-module. Then: \( R \) is a Noetherian \( R \)-module if, and only if \( R \) is a Noetherian ring.
2. Recall: Subrings of a Noetherian ring need not be Noetherian.
3. A finite direct sum of Noetherian modules is Noetherian.
(Hint: Induct on the number of summands & use Corollary 2.)
4. \( M \): Noetherian \( R \)-module \( \Rightarrow S^t M \) is a Noetherian \( S^t R \)-module.
\( S \) multi closed set of \( R \).

Proposition: Let \( R \) be a Noetherian ring. & \( M \) an \( R \)-module. Then 
\( M \) is Noetherian if, and only if \( M \) is finitely generated, i.e. 
\( \exists x_1, \ldots, x_e \in M \) st. every \( x \in M \) can be written (necessarily uniquely) as 
\( x = q_1 x_1 + \cdots + q_e x_e \) for \( q_1, \ldots, q_e \in R \).
Proof \((\Rightarrow)\) View \(M\) as a submodule of \(R\) & use Thm 3.

\((\Leftarrow)\) As \(M\) is finitely generated e.g. \(M = \langle x_1, \ldots, x_n \rangle\) we have \(R \xrightarrow{\phi} M\) morphism of \(R\)-modules. By the

universal property of \(\bigoplus_{i=1}^n R\) we have a unique

\[ f : \bigoplus_{i=1}^n R \longrightarrow M \quad \text{R-linear map s.t.} \quad \bigoplus_{i=1}^n a_i x_i \mapsto \sum_{i=1}^n a_i x_i \]

Furthermore, we get a ses of \(R\)-modules:

\[ 0 \longrightarrow \ker f \longrightarrow \bigoplus_{i=1}^n R \longrightarrow M \longrightarrow 0 \]

But \(R\) is Noetherian, so \(\bigoplus_{i=1}^n R\) is also a Noetherian \(R\)-module. Again by Corollary 2: \(\text{Im} f = M\) is Noetherian \(\square\).

**Examples**

1. \(R = \mathbb{K}\) a field, \(M = \mathbb{K}\)-vector space.

   \(M\) Noetherian \(\iff\) \(\dim_{\mathbb{K}} M < \infty\)

2. \(R = \mathbb{K}[x]\) (Noetherian ring, since it's a PID).

   - \(M = \mathbb{K}[x, y]\) is not a Noetherian \(R\)-module
     - (The \(R\)-submodule generated by \(1, y, y^2, \ldots\) is not f.g.
   - \(M\) will be a Noetherian ring!

3. An example of non-Noetherian ring:

   \[ R = \{ \text{continuous } \mathbb{C} \text{-valued functions on } \mathbb{R} \} \]

   \(F_n = \left[ \frac{1}{n}, \frac{1}{n} \right], n \geq 1\) \(\) (nested chain of intervals with \(|F_n| \searrow 0\))

   \(\alpha_n = \{ f \in R \mid f|_{F_n} \equiv 0 \}\) is an ideal in \(R\).

   \(\alpha_1 \subseteq \alpha_2 \subseteq \alpha_3 \subseteq \cdots\) is a strictly increasing chain of ideals.

   So \(R\) is not noetherian.