Lecture 24: Noetherian rings & modules

§1 Noetherian Rings:

Recall A commutative ring R is Noetherian if it satisfies ACC (every ascending chain of ideals stabilizes)

Theorem: If R is a commutative ring, the following are equivalent:

(1) R is Noetherian

(2) Every nonempty set $\mathcal{S}$ of ideals of R has a maximal element (with respect to inclusion)

(3) Every ideal $\mathfrak{a} \subseteq R$ is finitely generated.

Corollary 1: (1) Principal ideal domains are Noetherian (eg $\mathbb{Z}$, $\mathbb{C}[x]$)

(2) If $f: A \longrightarrow B$ is a ring homomorphism with A, B commutative. Assume f is surjective (key!). If A is Noetherian, so is B.

(3) Rings of fractions of Noetherian rings are Noetherian. In particular, localizations preserve Noetherianness.

Proof (2): If $\mathfrak{b} \subset B$ is an ideal, $\mathfrak{a} = f^{-1}(\mathfrak{b}) \subset A$ is an ideal, so $\mathfrak{a} = (a_1, \ldots, a_n)$. Then $\mathfrak{b} \underset{f \text{ surj}}{=} f(\mathfrak{a}) = (f(a_1), \ldots, f(a_n))$.

⚠ Subrings of Noetherian rings need not be Noetherian.

Ex: Take $R = \mathbb{C}[x_1, x_2, \ldots] = \bigcup_{n \in \mathbb{N}} \mathbb{C}[x_1, x_2, \ldots, x_n]$

Note: R is not Noetherian since the ascending chain of ideals

$$\mathfrak{a}_1 = (x_1) \subsetneq \mathfrak{a}_2 = (x_1, x_2) \subsetneq \ldots$$ never terminates.

- Now: R is a domain & $R \hookrightarrow \mathrm{Quot}(R) =$ field.

so R is a subring of $\mathrm{Quot}(R)$

- Since the only ideals of $\mathrm{Quot}(R)$ are (0) & (1), we get that $\mathrm{Quot}(R)$ is Noetherian.

Hilbert Basis Thm: If R is Noetherian, so is $R[x]$.

Hence $\mathbb{Z}[x_1, \dots, x_n]$, $\mathbb{K}[x_1, \dots, x_n]$ are Noetherian for any field $\mathbb{K}$.

To prove this result, we'll need the notion of Noetherian modules.

§2. Noetherian modules:

Fix R = commutative ring. & M an R-module.

Def: We say M is Noetherian if it satisfies the ascending chain condition for submodules:

"Every chain of submodules $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ stabilizes
ie $\exists\ l \geq 0$ st $M_l = M_{l+1} = \dots$"

We have the following analog of Theorem 1:

Theorem 2: Fix R a commutative ring & M an R-module. TFAE:

(1) M is Noetherian

(2) Every nonempty set $\mathcal{S}$ of submodules of M has a maximal element (with respect to inclusion)

(3) Every submodule of M is finitely generated.

The proof is exactly the same as that of Thm 1.

Corollary 2: Let $0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$ be a ses of R-modules. Then: M_2 is Noetherian if, and only if, M_1 & M_3 are.

Proof: $\Rightarrow$) submodules of M_1 are submodules of M_2 via f.

$N_3 \subseteq M_3$ come from $g^{-1}(N_3) \subseteq M_2$ submodules

$g^{-1}(N_3) = \langle m_1, \dots m_l \rangle$, then $N_3 = \langle g(m_1), \dots, g(m_l) \rangle$ by surjectivity of g.

$\Leftarrow$) Pick N a submodule of M_2 then $g(N) \subseteq M_3$ is a subm.

So $g(N) = \langle m_1, \dots, m_s \rangle$. Pick $n_1, \dots, n_s \in N$ with $g(n_1) = m_1, \dots, g(n_s) = m_s$.

Next take $f^{-1}(N \cap f(M_1)) = N_1 \subseteq M_1$ submodule, so its finitely generated $N_1 = \langle q_1, \dots q_\ell \rangle$

Then $n'_1 = f(q_1), \dots, n'_\ell = f(q_\ell)$. & $N \cap f(M_1) = \langle n'_1, \dots, n'_\ell \rangle$

Claim: $N = \langle n_1, \dots, n_s, n'_1, \dots, n'_\ell \rangle$

Pf/ Pick $n \in N$, so $g(n) \in g(N) = \langle m_1, \dots, m_s \rangle$

That is $g(n) = a_1 m_1 + \dots + a_s m_s = a_1 g(n_1) + \dots + a_s g(n_s)$
$= g(a_1 n_1 + \dots + a_s n_s)$ $\quad a_1, \dots a_s \in R$

This means $n - a_1 n_1 - \dots - a_s n_s \in \operatorname{Ker} g = \operatorname{Im} f$, so
$n - a_1 n_1 - \dots - a_s n_s \in N \cap f(M_1) = \langle n'_1, \dots, n'_\ell \rangle$

Conclude $n \in \langle n_1, \dots n_s, n'_1, \dots n'_\ell \rangle$ □

Note: (1) R is an R-module. Then: R is a Noetherian R-module if, and only if R is a Noetherian ring.

(2) Recall: Subrings of a Noetherian ring need not be Noetherian

(3) A finite direct sum of Noetherian modules is Noetherian

(Hint: Induct on the number of summands & use Corollary 2)

(4) M: Noetherian R-module $\Rightarrow S^{-1}M$ is a Noetherian $S^{-1}R$-mod. S mult closed set of R

Proposition: Let R be a Noetherian ring & M an R-module. Then M is Noetherian if & only if M is finitely generated, ie $\exists\, x_1, \dots, x_\ell \in M$ st every x in M can be written (no necessarily uniquely) as $x = a_1 x_1 + \dots + a_\ell x_\ell$ for $a_1, \dots, a_\ell \in R$.

Proof ($\Rightarrow$) View M as a submodule of M & use Theorem 2.

($\Leftarrow$) As M is finitely generated eg $M = \langle x_1, \dots, x_\ell \rangle$ we have

$$\begin{aligned} R &\xrightarrow{f_i} M \\ a_i &\longmapsto a_i x_i \end{aligned}$$

morphism of R-modules. By the universal property of $\underbrace{R \oplus \cdots \oplus R}_{\ell \text{ copies}}$ we have a unique

$$\begin{aligned} f : \bigoplus_{i=1}^{\ell} R &\longrightarrow M \\ (a_1, \dots, a_\ell) &\longmapsto \sum_{i=1}^{\ell} a_i x_i \end{aligned}$$

R-linear map st

$$\begin{tikzcd} \bigoplus_{i=1}^{\ell} R \arrow[drr, "f"] & & \\ R \arrow[u, "j_i"] \arrow[rr, "f_i"'] \arrow[urr, phantom, "\circlearrowleft"] & & M \end{tikzcd} \quad \forall i$$

Furthermore, we get a ses of R-modules:

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow \bigoplus_{i=1}^{\ell} R \longrightarrow M \longrightarrow 0$$

But R is Noetherian, so $\bigoplus_{i=1}^{\ell} R$ is also a Noetherian R-module. Again by Corollary 2: $\operatorname{Im} f = M$ is Noetherian $\square$.

Examples ① $R = \mathbb{K}$ a field, $M = \mathbb{K}$-vector space

M Noetherian $\Longleftrightarrow$ $\dim_{\mathbb{K}} M < \infty$

② $R = \mathbb{K}[x]$ (Noetherian ring, since it's a PID)

- $M = \mathbb{K}[x,y]$ is not a Noetherian R-module
  (The R-submodule generated by $\{1, y, y^2, \dots\}$ is NOT f.g.
- M will be a Noetherian ring!

③ An example of non-Noetherian ring:

$R = \{$continuous $\mathbb{C}$-valued functions on $\mathbb{R}\}$

$F_n = [-\frac{1}{n}, \frac{1}{n}]$ $n \geq 1$ (nested chain of intervals with $|F_n| \searrow 0$)

$\mathfrak{a}_n = \{f \in R \mid f|_{F_n} \equiv 0\}$ is an ideal in R.

$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \cdots\cdots$ is a strictly increasing chain of ideals.

So R is Not noetherian.