Lecture 25: Hilbert Basis Theorem & Artinian Rings

Last Time: discussed Noetherian modules (ACC, submodules are f.g)

TODAY: • Proof of Hilbert Basis Theorem
• Artinian rings, defined using descending chains.

§1 Hilbert Basis Theorem:

Theorem: If R is commutative and Noetherian, so is $R[X]$.

Proof: We will show that every ideal of $R[X]$ is finitely generated.
Let $\mathfrak{b} \subset R[X]$ be an ideal. For every $f(x) \in R[X]$ we let $L.T(f) \in R$ be the leading coefficient of f

$0 \neq f = a_0 + a_1 X + \cdots + a_n x^n$ with $a_n \neq 0 \rightsquigarrow LT(f) := a_n \in R$

We define $LT(0) = 0$.

Claim: $\mathfrak{a} = \{LT(f) : f \in \mathfrak{b}\} \subset R$ is an ideal.

Pf/ (1) $0 \in \mathfrak{a}$ since $LT(0) = 0$ & $0 \in \mathfrak{b}$

(2) $a\, LT(f) \in \mathfrak{a}$ $\forall a \in R$ & $f \in \mathfrak{b}$. Clear if $a LT(f) = 0$

Otherwise $a LT(f) = LT(af) \in \mathfrak{a}$ $\quad \left(a\sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} (a a_i) x^i \right)$
$a_n \neq 0$ & $a a_n \neq 0$.

So $-LT(f) \in \mathfrak{a}$ if $f \in \mathfrak{b}$.

(3) $LT(f) + LT(g) \in \mathfrak{a}$ if $LT(f)$ & $LT(g)$ do

• If $LT(f) = -LT(g)$ we know $0 \in \mathfrak{a}$, so assume

$LT(f) + LT(g) \neq 0$. & both $f, g \neq 0$, so they have a degree.

• We can assume $\underset{k}{\underbrace{\deg(f)}} \leq \underset{l}{\underbrace{\deg g}}$. Then, $x^{l-k} f \in \mathfrak{b}$ &
$g \in \mathfrak{b}$

so $LT(f) + LT(g) \underset{\substack{\uparrow \\ \text{no cancellation occurs}}}{=} LT(\underbrace{x^{l-k} f + g}_{\in \mathfrak{b} \text{ (since } f \& g \in \mathfrak{b})}) \in \mathfrak{a}$

Since R is Noetherian, we know $\mathfrak{a}$ is f.g. by $a_1, \ldots a_\ell$ with $a_i \neq 0 \ \forall i$. For each $j = 1, \ldots, \ell$ pick $f_j \in \mathfrak{b}$ with $\boxed{a_j = LT(f_j).}$

Let $r = \max\limits_{1 \leq j \leq \ell} \deg(f_j) \geq 0$. Let $M \subset R[x]$ be the R-submodule generated by $\{1, x, \ldots, x^{r-1}\}$ (so M is the set of polynomials of degree $< r$) Since R is Noetherian & M is f.g, then M is noetherian.

Now $\mathfrak{b} \cap M \subset M$ is a submodule of M so it's also finitely generated say by $\{b_1, \ldots, b_k\}$.

<u>Claim 2</u>: $\mathfrak{b} = \langle b_1, \ldots, b_k, f_1, \ldots f_\ell \rangle$

Pf/ Pick $f \in \mathfrak{b}$. If $\deg(f) < r$, then $f \in \mathfrak{b} \cap M$ & hence $f \in \langle b_1, \ldots, b_k \rangle$. Otherwise, we proceed by induction on $\deg(f) \geq r$.

Let $a = LT(f)$ with $\deg(f) = d \geq \deg(f_j) =: d_j \ \forall j$

Since $a \in \mathfrak{a}$, we have $a = r_1 a_1 + \cdots + r_\ell a_\ell$ for suitable $r_1, \ldots r_\ell$

Thus, $g = f - \sum r_j x^{d-d_j} f_j \in \mathfrak{b}$ & $\deg g < \deg f$.

- If $\deg f < r$, then $g \in \mathfrak{b} \cap M$ and we are done: Indeed:

$$g = f - \sum r_j x^{d-d_j} f_j = c_1 b_1 + \cdots + c_k b_k$$

so $f \in \langle b_1, \ldots b_k, f_1, \ldots, f_\ell \rangle$

- If $\deg f \geq r$, then $\deg g < \deg f$ & $g \in \mathfrak{b}$. By IH, $g \in \langle b_1, \ldots, b_k, f_1, \ldots, f_\ell \rangle$, so the same holds for f. $\square$

§2 Artinian rings: Definition & first properties

Q Why study Artinian rings?

A: Geometrically Artinian rings correspond to finite collections of fat points (ie, points with multiplicities)

(• Str. Thm $R \simeq R/\mathfrak{m}_1^n \times \cdots \times R/\mathfrak{m}_\ell^n$
R Art. $\mathfrak{m}_i$ maxl ideals $i = 1, \ldots, \ell$
• Noeth + dim 0 $\Leftrightarrow$ Artinian)

Definition: Let R be a commutative ring. We say that R is Artinian (after Emil Artin) if every descending chain of ideals $\mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \ldots$ stabilizes, ie $\exists\, \ell \geq 0$ with $\mathfrak{a}_\ell = \mathfrak{a}_{\ell+1} = \cdots$ (Descending Chain Condition)

Ex: ① $R = \mathbb{K}$ field is Artinian

② $R = \mathbb{K}[x]/(x^n)$ ——— (Ideals are $\mathbb{K}$-subspaces & $\dim_{\mathbb{K}} R = n$)

Lemma 1: Let $\mathcal{S}$ be non-empty set of ideals in an Artinian ring. Then $\mathcal{S}$ has minimal elements (with respect to inclusion)

Proof: (Same idea as for Noetherian rings)

Let $\mathfrak{a}_0 \in \mathcal{S}$. If $\mathfrak{a}_0$ is minimal, we are done. Otherwise, we find $\mathfrak{a}_1 \in \mathcal{S}$ with $\mathfrak{a}_0 \supsetneq \mathfrak{a}_1$. As R is Artinian, this process must stop and we will arrive at a minimal element of $\mathcal{S}$. □

Lemma 2: Artinian property is preserved under quotients by ideals

Pf/ Let $\mathfrak{a} \subseteq R$ be an ideal and R be Artinian

Then $\tilde{R} = R/\mathfrak{a}$ is also Artinian since ideals in $\tilde{R}$ correspond to ideals in R containing $\mathfrak{a}$.

So the DCC for R yields the DCC for $\tilde{R}$. □

Proposition 1: Let R be an Artinian commutative ring. Then:

(i) Every prime ideal in R is maximal.

(ii) There are only finitely many maximal ideals in R.

Proof: (i) Let $\mathfrak{p} \subsetneq R$ be a prime ideal. Then $R/\mathfrak{p}$ is an Artinian integral domain. (by Lemma 2)

Now that $x \in R/\mathfrak{p} \setminus \{0\}$ & consider the descending chain of ideals in $R/\mathfrak{p}$:

$$(x) \supseteq (x^2) \supseteq (x^3)$$

Since it eventually stabilizes, $\exists\, k \geq 1$ with $(x^k) = (x^{k+1})$

ie $x^k = y x^{k+1}$ for $y \in R/\mathfrak{p}$.

$\Rightarrow x^k(1 - xy) = 0$

As $R/\mathfrak{p}$ is a domain and $x \neq 0$ we have $1 = xy$, so x is a unit.

We conclude $(R/\mathfrak{p})^\times = R/\mathfrak{p} \setminus \{0\}$, so $R/\mathfrak{p}$ is a field

This means $\mathfrak{p}$ is a maximal ideal of R. $\square$

(ii) Let $\boxed{\mathcal{J} = \text{set of ideals that are intersections of finitely many maximal ideals of } R}$

- $\mathcal{J} \neq \emptyset$ since maximal ideals of R exist & lie in $\mathcal{J}$.
- By the Artinian condition & Lemma 1, $\mathcal{J}$ has a minimal element $\mathfrak{a} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell$

Claim: $\{\text{Maximal ideals in } R\} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell\}$

Pf/ Pick $\mathfrak{m} \subsetneq R$ maximal ideal, then $\mathfrak{m} \cap \mathfrak{a} \in \mathcal{J}$ & $\mathfrak{m} \cap \mathfrak{a} \subseteq \mathfrak{a}$. By the minimality of $\mathfrak{a}$, we have $\mathfrak{a} = \mathfrak{m} \cap \mathfrak{a}$, so $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell = \mathfrak{m} \cap \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_\ell \subseteq \mathfrak{m}$ (prime)

By Prime Avoidance (Lecture 20 Thm 2, page 4) $\exists\, j$ st $\mathfrak{m}_j \subseteq \mathfrak{m}$

Since $\mathfrak{m}_j$ & $\mathfrak{m}$ are both maximal we have $\mathfrak{m}_j = \mathfrak{m}$. $\square$