Lecture 26: Artinian rings II

Last time: We defined Artinian rings \( R \) as rings with DCC
(Equivalently: Every non-empty set of ideals of \( R \) has a minimal element)

- Quotients of Artinian rings are Artinian.

**Proposition 1**: Let \( R \) be an Artinian commutative ring. Then:

1. Every prime ideal in \( R \) is maximal.
2. There are only finitely many maximal ideals in \( R \)

**Property**: \( R \) Artinian and an integral domain \( \Rightarrow \) \( R \) is a field (from the proof of (ii))

**Corollary**: Nilradical = Jacobson radical for Artinian rings

**Proposition 2**: The nilradical ideal \( N \subseteq R \) (Artinian) is nilpotent, i.e., \( \exists \ n \geq 0 \) such that \( N^n = (0) \).

**Proof**: See HW 9.

**Examples**: 
1. \( \mathbb{K} \) \((N = (0), s_{0,1})\)
2. \( \mathbb{K}[x]/(x^n) \) \((N = (x)) \quad so \quad N^n = (0)\)

**Today**: Structure Theorem of Artinian rings
- Artinian local rings are Noetherian.

**3.1. Structure Theorem**:

Structure Theorem = Geometric interpretation of Artinian Rings

- Finite collections of fat points.

We will need the following general result:

**Lemma**: Fix \( R \) a commutative ring and \( \alpha, \beta \subseteq A \) be two coprime ideals. Then

1. \( \alpha^i \cap \beta^i \) are coprime \( \forall i \geq 1 \) \((\alpha + \beta = (1))\)
2. \( \alpha \cdot \beta = \alpha \cap \beta \)

(In particular: \( \alpha^i \cdot \beta^i = \alpha \cap \beta^i \) for all \( i \geq 1 \))
Remark: Item (2) generalizes to a finite collection of pairwise coprime ideals (see HW9).

Proof of Lemma: Since \( \alpha + \beta = 1 \), \( \exists \ a \in \alpha \ \& \ b \in \beta \) with \( 1 = a + b \)

\[ 1 = (a + b)^{2i} = \sum_{j=0}^{2i} \binom{2i}{j} a^{2i-j} b^j \]

So \( 1 = \left( \sum_{j=0}^{i} \binom{2i}{j} b^j a^{2i-j} \right) a^i + \left( \sum_{j=1}^{i} \binom{2i}{j+i} a^{2i-j} b^j \right) b^i \in \alpha^i \in \beta^i \)

(2) \( \alpha \cdot \beta \subseteq \alpha \cap \beta \) is always true

For the claim, pick \( x \in \alpha \cap \beta \):

\[ x = a + b \quad \Rightarrow \quad x = a \cdot b \in \alpha \beta \]

From now on, we fix \( R = \) commutative & Artinian ring.

We let \( \begin{bmatrix} m_1, \ldots, m_e \end{bmatrix} \) be the maximal ideals (=prime ideals) of \( R \)

Note that maximal ideals are always pairwise coprime

By construction, \( N = \cap m \ldots m_e \) is a nilpotent ideal, so \( \exists n \) with \( N^n = 0 \)

Structure Theorem: With the above conditions \( m R, l \ & n \), we have

\[ R \cong R_{m1}^m \times \cdots \times R_{m_e}^m \] (ring isomorphism)

Furthermore, each \( R_{m_j}^m \) is Artinian and local with unique max ideal \( \bar{m} \).

Proof: Since \( m_1, \ldots, m_e \) are pairwise coprime, we can use the CRT.

Consider \( R \xrightarrow{\Phi} R_{m_1}^m \times \cdots \times R_{m_e}^m \) where \( \Phi_j : R \rightarrow R_{m_j}^m \) is the natural projection.
By construction, $\psi$ is a ring homomorphism.

By CRT, $\psi$ is surjective.

$\text{Ker} \, \psi = m_1^n \cap \cdots \cap m_k^n = m_1^n \cdots m_k^n \subseteq (m_1, \ldots, m_k) \cong \mathbb{Z}^k$.

Remark: $(m_i^n)$ pairwise coprime holds for commutative rings.

Since $m_1^n \cdots m_k^n = m_1 \cap \cdots \cap m_k = \mathbb{N}$

Conclusion: $\psi$ is a ring isomorphism.

It remains to show that $R/m_i^n$ is local $1 \leq i \leq k$.

Let $\mathfrak{q} \not\subseteq R/m_i^n$ be a maximal ideal, so it's prime.

Then $\mathfrak{q}$ is the image of $\mathfrak{p} = \psi^{-1}(\mathfrak{q}) \subseteq R$ & $\mathfrak{p}$ is a prime ideal containing $m_i^n$.

Problem 7.44.7: $m_i^n \subseteq \mathfrak{p}$, $\mathfrak{p}$ maximal & $\mathfrak{p}$ prime $\Rightarrow m_i^n = \mathfrak{p}$.

We conclude $m_i = \mathfrak{p}$ & so $\mathfrak{q} = \psi(m_i) = m_i/m_i^n$.

5.2 The local case:

Proposition: If $R$ is Artinian and local, then $R$ is Noetherian.

Combining this proposition with The Structure Theorem for Artinian rings & the fact that finite products of Noetherian rings are Noetherian, we get.

Corollary: Artinian rings are Noetherian & have dimension 0.

Corollary: Next time!

Proof of Proposition: Let $M$ be the unique maximal ideal of $R$ & write $k = R/M$ for the quotient. We know $k$ is a field.
Note: For each \( j \geq 0 \), the set \( \frac{M_j}{m_{j+1}} \) is an \( R \)-module (it's a quotient of \( R \)-modules).

Now, \( M \cdot m_j \subseteq m_{j+1} \) so \( M \cdot \frac{m_j}{m_{j+1}} = (0) \). This says that \( \frac{m_j}{m_{j+1}} \) is in fact an \( R \)-module, i.e., a \( k \)-vector space.

Claim: \( R \) Artinian \( \Rightarrow \text{dim}_k \frac{m_j}{m_{j+1}} < \infty \). For all \( j \geq 0 \)

(We'll see the proof of this next time).

Assuming the claim holds, we will prove that \( R \) is Noetherian.

Pick an ideal \( \mathcal{A} \) of \( R \). We want to show \( \mathcal{A} \) is finitely generated.

We need to construct subspaces of \( \frac{m_j}{m_{j+1}} \) from each \( j \).

For each \( j \geq 0 \), we consider the ideal \( \mathcal{A}_j = \mathcal{A} \cap m_j \) of \( R \).

We consider the following maps of \( R \)-modules:

\[
\mathcal{A}_j \hookrightarrow m_j \xrightarrow{\pi} \frac{m_j}{m_{j+1}}
\]

By construction, \( f \) is \( R \)-linear.

\( \ker(f) = \mathcal{A}_j \cap m_{j+1} = \mathcal{A} \cap m_j \cap m_{j+1} = \mathcal{A} \cap m_{j+1} = \mathcal{A}_{j+1} \).

So \( \bar{f} : \frac{\mathcal{A}_j}{\mathcal{A}_{j+1}} \longrightarrow \frac{m_j}{m_{j+1}} \) is injective \& \( R \)-linear.

Now \( M \cdot \frac{\mathcal{A}_j}{\mathcal{A}_{j+1}} = (0) \) \& \( M \cdot \frac{m_j}{m_{j+1}} \), so \( \bar{f} \) can be viewed as a \( k \)-linear injective map, that is, as an inclusion of vector spaces.

Hence: \( \frac{\mathcal{A}_j}{\mathcal{A}_{j+1}} \) is a \( k \)-subspace of \( \frac{m_j}{m_{j+1}} \) (via \( \bar{f} \)), thus, finite dimensional by the claim.

Since \( M^n = (0) \), we have \( A_n = (0) \), so we only need to look at \( j = 0, \ldots, n \).
For each \( j = 0, \ldots, n-1 \) with \( \alpha_j \neq (0) \), we let \( \tilde{\alpha}_j = (\alpha_j(1), \ldots, \alpha_j(n)) \) be a \( k \)-basis for \( \alpha_j \), with \( \alpha_j(1), \ldots, \alpha_j(n) \in \alpha_j(1) \). To simplify notation, we set \( \alpha_j = (0) \) if \( \alpha_j = (0) \). We'll use these bases to build a generating set for \( \alpha \)

Write \( \tilde{\alpha} := \bigcup_j \tilde{\alpha}_j \) a set \( \tilde{\alpha} \) be the ideal generated by \( \tilde{\alpha} \) (so \( \tilde{\alpha} \) is \( f.g \)).

**Claim 2:** \( \tilde{\alpha} = \alpha \), so \( \alpha \) is \( f.g \).

By construction, we have \( \tilde{\alpha} \subseteq \alpha \). To prove the claim it's enough to show \( \tilde{\alpha}_j = \alpha_j \) for all \( j = 0, \ldots, n \) where \( \tilde{\alpha}_j := \tilde{\alpha} \cap \mathfrak{m}^j \) for all \( j \).

Taking \( j = 0 \) will give \( \tilde{\alpha} = \tilde{\alpha} \cap \mathfrak{m} = \tilde{\alpha} = \alpha \cap \mathfrak{m} = \alpha \).

- We show \( \tilde{\alpha}_j = \alpha_j \) for \( j = 0, \ldots, n \) by reverse induction on \( j \).

**Base case:** \( j = n \) \( \tilde{\alpha}_n = \alpha_n = (0) \) since \( \mathfrak{m}^n = (0) \).

**Inductive step:** Assume \( j < n \) and that the result holds for \( j+1 \).

Notice that we have the following inclusions of ideals of \( R \):

\[
\tilde{\alpha}_j := \tilde{\alpha} \cap \mathfrak{m}^j \subseteq \alpha_j := \alpha \cap \mathfrak{m}^{j+1}
\]

As above, we consider the composition of \( R \)-linear maps:

\[
\tilde{\alpha}_j \longrightarrow \alpha_j \overset{\pi}{\longrightarrow} \alpha_j/\alpha_j^{j+1}
\]

- \( \pi \) is \( R \)-linear = \( f \).
- \( \ker f = \tilde{\alpha} \cap \alpha_j = \tilde{\alpha} \cap \alpha_j \cap \mathfrak{m}^{j+1} = (\tilde{\alpha} \cap \alpha_j) \cap (\mathfrak{m}^j \cap \mathfrak{m}^{j+1}) = \tilde{\alpha} \cap \mathfrak{m}^{j+1} = \tilde{\alpha}_j^{j+1} \).

By 1st isom theorem, we get an \( R \)-linear injection map \( \tilde{f} : \tilde{\alpha}_j/\tilde{\alpha}_j^{j+1} \longrightarrow \alpha_j/\alpha_j^{j+1} \).
As before, \( m \cdot \alpha_j / \bar{\alpha}_{j+1} = m \cdot \tilde{\alpha}_j / \bar{\alpha}_{j+1} = (0) \), so \( \bar{\alpha}_{j+1} \) is \( k \)-linear.

Using this, we get an inclusion of \( k \)-vector spaces \( \tilde{\alpha}_j / \bar{\alpha}_{j+1} \rightarrow \alpha_j / \bar{\alpha}_{j+1} \).

Notice that the basis \( \{ \tilde{\alpha}_j^{(b)}, \ldots, \tilde{\alpha}_j^{(e)} \} \) of \( \tilde{\alpha}_j / \bar{\alpha}_{j+1} \) consists of left-cosets with representatives \( \tilde{\alpha}_j^{(b)} \) in \( \tilde{\alpha}_j \), so this inclusion is an equality! Thus \( \tilde{\alpha}_j = \alpha_j / \bar{\alpha}_{j+1} \).

In particular, \( \alpha_j = \tilde{\alpha}_j + \alpha_{j+1} = \tilde{\alpha}_j + \alpha_{j+1} = \tilde{\alpha}_j \), as we wanted to show.

Next time, we'll prove the claim by showing

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**Lemma 2.** If \((R, M)\) is Artinian local, then \( \dim \frac{M_j}{M_{j+1}} < \infty \) for all \( j \).

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