Lecture 26: Artinian rings II

Last time: We defined **Artinian** rings R as rings with DCC
(Equivalently: Every non-empty set of ideals of R has a minimal element)

- Quotients of Artinian rings are Artinian.

Proposition 1: Let R be an Artinian commutative ring. Then:
(i) Every prime ideal in R is maximal.
(ii) There are only finitely many maximal ideals in R

Property: R Artinian and an integral domain $\Rightarrow$ R is a field.
(from the proof of (i))

Corollary: Nilradical = Jacobson radical for Artinian rings
$\quad\|$ Pr 2 HW8 $\qquad\qquad \|$ Pr 8 HW7
$\bigcap$ prime ideals $\qquad \bigcap$ mxl ideals

Proposition 2: The nilradical ideal $\mathcal{N} \subset R$ (Artinian) is nilpotent,
ie $\exists\, n \geq 0$ such that $\mathcal{N}^n = (0)$.
Proof: See HW9.

Examples: ① $\mathbb{K}$ $\;(\mathcal{N} = (0)$, so $n=1)$; ② $\mathbb{K}[x]/(x^n)$. $\;\left(\mathcal{N} = \frac{(x)}{(x^n)} \text{ so } \mathcal{N}^n = (0)\right)$

TODAY:
- Structure Theorem of Artinian rings
- Artinian local rings are Noetherian.

§1. Structure Theorem:

Structure Theorem = Geometric interpretation of Artinian Rings
= finite collections of fat points.

- We will need the following general result:

Lemma 1: Fix R a commutative ring and $\mathfrak{a}, \mathfrak{b} \subset A$ be two coprime ideals. Then
$(\mathfrak{a} + \mathfrak{b} = (1))$
(1) $\mathfrak{a}^i$ & $\mathfrak{b}^i$ are coprime $\forall i \geq 1$.
(2) $\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{a} \cap \mathfrak{b}$

(In particular: $\mathfrak{a}^i \cdot \mathfrak{b}^i = \mathfrak{a}^i \cap \mathfrak{b}^i$ for all $i \geq 1$)

<u>Remark</u>: Item (2) generalizes to a finite collection of pairwise coprime ideals (see HW9)

<u>Proof of Lemma</u>: Since $\mathfrak{a}+\mathfrak{b}=1$, $\exists\, a\in\mathfrak{a}$ & $b\in\mathfrak{b}$ with $1=a+b$

(1) $1=(a+b)^{2i}=\sum_{j=0}^{2i}\binom{2i}{j}a^{2i-j}b^j$

$$\text{So}\quad 1=\underbrace{\left(\sum_{j=0}^{i}\binom{2i}{j}b^j a^{i-j}\right)a^i}_{\in\,\mathfrak{a}^i}+\underbrace{\left(\sum_{j=1}^{i}\binom{2i}{j+i}a^{i-j}b^j\right)b^i}_{\in\,\mathfrak{b}^i}$$

(2) $\mathfrak{a}\cdot\mathfrak{b}\subseteq\mathfrak{a}\cap\mathfrak{b}$ is always true

For the converse, pick $x\in\mathfrak{a}\cap\mathfrak{b}$:

$$1=a+b\quad\text{so}\quad x=\underbrace{a\overset{\in\mathfrak{b}}{x}}_{\in\mathfrak{a}\mathfrak{b}}+\underbrace{\overset{\in\mathfrak{a}}{x}b}_{\in\mathfrak{a}\mathfrak{b}}\in\mathfrak{a}\mathfrak{b}\qquad\square$$

- From now on, we fix R = commutative & Artinian ring.
- We let $\boxed{\mathfrak{m}_1,\dots,\mathfrak{m}_\ell}$ be the <u>maximal ideals</u> (= prime ideals) of R

  Note that maximal ideals are always pairwise coprime
- By construction, $\mathcal{N}=\mathfrak{m}_1\cap\dots\cap\mathfrak{m}_\ell$ is a nilpotent ideal, so $\exists\, n\geq 1$ with $\mathcal{N}^n=(0)$

<u>Structure Theorem</u>: With the above conditions on R, ℓ & n, we have

$$R\simeq R/\mathfrak{m}_1^n\times\dots\times R/\mathfrak{m}_\ell^n\qquad\text{(ring isomorphism)}$$

Moreover, each $R/\mathfrak{m}_j^n$ is Artinian and local with unique max'l ideal $\mathfrak{m}_j/\mathfrak{m}_j^n$.

<u>Proof</u>: Since $\mathfrak{m}_1^n,\dots,\mathfrak{m}_\ell^n$ are pairwise coprime, we can use the CRT.

Consider $R\xrightarrow{\varphi}R/\mathfrak{m}_1^n\times\dots\times R/\mathfrak{m}_\ell^n$, $x\longmapsto(\pi_1(x),\dots,\pi_\ell(x))$ where $\pi_j:R\longrightarrow R/\mathfrak{m}_j^n$ is the natural projection

- By construction, φ is a ring homomorphism.
- By CRT, φ is surjective
- $\operatorname{Ker}\varphi = m_1^n \cap \cdots \cap m_\ell^n \underset{\substack{\uparrow \\ \text{Remark} \\ (m_i^n \text{ pairwise coprime})}}{=} m_1^n \cdots m_\ell^n \underset{\substack{\uparrow \\ \text{holds for commutative rings}}}{\subseteq} (m_1 \cdots m_\ell)^n \subseteq \mathcal{N}^n = (0)$

since $m_1 \cdots m_\ell = m_1 \cap \cdots \cap m_\ell = \mathcal{N}$

Conclusion: φ is a ring isomorphism.

- It remains to show that R/m_j^n is local $\forall\, 1 \leq j \leq \ell$.

Let $q \subsetneq R/m_j^n$ be a maximal ideal, so it's prime

Then q is the image of $p = \pi^{-1}(q) \subseteq R$ & p is a prime ideal containing m_j^n.

- Problem 7 HW7: $m^n \subseteq p$ m maximal & p prime $\Rightarrow m = p$.

We conclude $m_j = p$ & so $q = \pi(m_j) = m_j/m_j^n$. $\square$

§2 The local case:

Proposition: If R is Artinian and local, then R is Noetherian

Combining this Proposition with The Structure Theorem for Artinian rings & the fact that finite products of Noetherian rings are Noetherian, we get.

Corollary Artinian rings are Noetherian & have dimension 0 (Prop 1 (1))

Converse: Next Time!

Proof of Proposition: Let m be the unique maximal ideal of R & write $k = R/m$ for the quotient. We know k is a field.

<u>Note</u>: For each $j \geq 0$: the set $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is an R-module (it's a quotient of R-modules).

Now, $\mathfrak{m} \cdot \mathfrak{m}^j \subseteq \mathfrak{m}^{j+1}$ so $\mathfrak{m} \cdot \frac{\mathfrak{m}^j}{\mathfrak{m}^{j+1}} = (0)$. This says that $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is in fact an $R/\mathfrak{m}$-module, ie a k-vector space.

<u>Claim</u>: R Artinian $\Longrightarrow$ $\boxed{\dim_k \mathfrak{m}^j/\mathfrak{m}^{j+1} < \infty}$. for all $j \geq 0$

(We'll see the proof of this next time).

- Assuming the claim holds, we will prove that R is Noetherian.

Pick an ideal $\mathfrak{a}$ of R. We want to show $\mathfrak{a}$ is finitely generated. We need to construct subspaces of $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ from each j.

For each $j \geq 0$, we consider the ideal $\boxed{\mathfrak{a}_j = \mathfrak{a} \cap \mathfrak{m}^j}$ of R

We consider the following maps of R-modules:

$$\begin{tikzcd} \mathfrak{a}_j \arrow[r, hook] \arrow[rr, bend right, "=:f"'] & \mathfrak{m}^j \arrow[r, "\pi"] & \mathfrak{m}^j/\mathfrak{m}^{j+1} \end{tikzcd}$$

- By construction, f is R-linear
- $\ker(f) = \mathfrak{a}_j \cap \mathfrak{m}^{j+1} = \mathfrak{a} \cap \mathfrak{m}^j \cap \mathfrak{m}^{j+1} = \mathfrak{a} \cap \mathfrak{m}^{j+1} = \mathfrak{a}_{j+1}$.

So $\bar{f} : \frac{\mathfrak{a}_j}{\mathfrak{a}_{j+1}} \hookrightarrow \mathfrak{m}^j/\mathfrak{m}^{j+1}$ is injective & R-linear.

- Now $\mathfrak{m} . \mathfrak{a}_j/\mathfrak{a}_{j+1} = (0)$ & $\mathfrak{m} \cdot \mathfrak{m}^j/\mathfrak{m}^{j+1}$ so $\bar{f}$ can be viewed as a k-linear & injective map, that is, as an inclusion of vector spaces.

<u>In short</u>: $\mathfrak{a}_j/\mathfrak{a}_{j+1}$ is a k-subspace of $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ (via $\bar{f}$), thus, finite dimensional by the claim.

Since $\mathfrak{m}^n = (0)$, we have $\mathfrak{a}_n = (0)$, so we only need to look at $j = 0, \ldots, n-1$

• For each $j=0,\dots,n-1$ with $\frac{\mathfrak{a}_j}{\mathfrak{a}_{j+1}} \neq (0)$, we let $\{\bar{a}_1^{(j)},\dots,\bar{a}_{\alpha_j}^{(j)}\}$ be a k-basis for $\mathfrak{a}_j/\mathfrak{a}_{j+1}$, with $a_1^{(j)},\dots,a_{\alpha_j}^{(j)} \in \mathfrak{a}_j$. To simplify notation, we set $\alpha_j = 0$ if $\frac{\mathfrak{a}_j}{\mathfrak{a}_{j+1}} = (0)$. We'll use these bases to build a generating set for $\mathfrak{a}$.

Write $\mathcal{B} := \bigcup_{j=1}^{n-1} \{a_1^{(j)},\dots,a_{\alpha_j}^{(j)}\}$ & let $\tilde{\mathfrak{a}}$ be the ideal generated by $\mathcal{B}$ (so $\tilde{\mathfrak{a}}$ is f.g.)

<u>Claim 2</u>: $\tilde{\mathfrak{a}} = \mathfrak{a}$, so $\mathfrak{a}$ is f.g.

By construction, we have $\tilde{\mathfrak{a}} \subseteq \mathfrak{a}$. To prove the claim it's enough to show $\tilde{\mathfrak{a}}_j = \mathfrak{a}_j$ for all $j=0,\dots,n$ where $\tilde{\mathfrak{a}}_j := \tilde{\mathfrak{a}} \cap \mathfrak{m}^j$ for all j.

Taking $j=0$ will give $\tilde{\mathfrak{a}} = \tilde{\mathfrak{a}} \cap R = \tilde{\mathfrak{a}}_0 = \mathfrak{a}_0 = \mathfrak{a} \cap R = \mathfrak{a}$

• We show $\tilde{\mathfrak{a}}_j = \mathfrak{a}_j$ for $j=0,\dots,n$. by reverse induction on j

• <u>Base case</u>: $j=n$ $\quad \tilde{\mathfrak{a}}_n = \mathfrak{a}_n = (0)$ since $\mathfrak{m}^n = (0)$

• <u>Inductive step</u>: Assume $j<n$ & that the result holds for $j+1$.

Notice that we have the following inclusions of ideals of R

$$\begin{array}{ccc}
\mathfrak{a}_j := \mathfrak{a} \cap \mathfrak{m}^j & \supseteq & \mathfrak{a}_{j+1} := \mathfrak{a} \cap \mathfrak{m}^{j+1} \\
\cup\!\mathsf{I} & & \cup\!\mathsf{I} \\
\tilde{\mathfrak{a}}_j := \tilde{\mathfrak{a}} \cap \mathfrak{m}^j & \supseteq & \tilde{\mathfrak{a}}_{j+1} := \tilde{\mathfrak{a}} \cap \mathfrak{m}^{j+1}
\end{array}$$

As above, we consider the composition of R-linear maps:

$$\begin{tikzcd}
\tilde{\mathfrak{a}}_j \arrow[r, hook] \arrow[rr, bend right, "=: f"'] & \mathfrak{a}_j \arrow[r, "\pi"] & \mathfrak{a}_j/\mathfrak{a}_{j+1}
\end{tikzcd}$$

• f is R-linear

• $\operatorname{Ker} f = \tilde{\mathfrak{a}}_j \cap \mathfrak{a}_{j+1} = \tilde{\mathfrak{a}} \cap \mathfrak{m}^j \cap \mathfrak{a} \cap \mathfrak{m}^{j+1} = (\tilde{\mathfrak{a}} \cap \mathfrak{a}) \cap (\mathfrak{m}^j \cap \mathfrak{m}^{j+1})$
$= \tilde{\mathfrak{a}} \cap \mathfrak{m}^{j+1} = \tilde{\mathfrak{a}}_{j+1}$.

By 1st iso Theorem, we get an R-linear injective map $\bar{f} : \tilde{\mathfrak{a}}_j/\tilde{\mathfrak{a}}_{j+1} \longrightarrow \mathfrak{a}_j/\mathfrak{a}_{j+1}$

As before $\mathfrak{m} \cdot \mathfrak{a}_j/\mathfrak{a}_{j+1} = \mathfrak{m}\,\tilde{\mathfrak{a}}_j/\tilde{\mathfrak{a}}_{j+1} = (0)$, so $\bar{F}$ is k-linear.

Using this we get an inclusion of k-vector spaces $\frac{\tilde{\mathfrak{a}}_j}{\tilde{\mathfrak{a}}_{j+1}} \hookrightarrow \mathfrak{a}_j/\tilde{\mathfrak{a}}_{j+1}$

Notice that the basis $\{\bar{a}_1^{(j)}, \dots, \bar{a}_{\alpha_j}^{(j)}\}$ of $\mathfrak{a}_j/\mathfrak{a}_{j+1}$ consists of left-cosets with representatives $a_i^{(j)}$ in $\tilde{\mathfrak{a}}_j$, so this inclusion is an equality! Thus $\frac{\tilde{\mathfrak{a}}_j}{\tilde{\mathfrak{a}}_{j+1}} = \frac{\mathfrak{a}_j}{\mathfrak{a}_{j+1}}$.

In particular: $\mathfrak{a}_j \subseteq \tilde{\mathfrak{a}}_j + \mathfrak{a}_{j+1} \underset{\text{IH}}{=} \tilde{\mathfrak{a}}_j + \tilde{\mathfrak{a}}_{j+1} = \tilde{\mathfrak{a}}_j$, as we wanted to show. $\square$

Next Time: we'll prove the claim, by showing

<u>Lemma 2</u>: If $(R, \mathfrak{m})$ is Artinian & local (commutative), then $\dim_{R/\mathfrak{m}} \mathfrak{m}^j/\mathfrak{m}^{j+1} < \infty$. $\forall j$.