Lecture 27: Artinian Rings & Primary Decomposition

Last Time: Discuss Artinian rings & local Artinian rings:

- **Structure Theorem**: If R is a commutative Artinian ring, then:
  (1) R has finitely many max'l ideals $(m_1, \ldots, m_e)$ $\quad (\mathcal{N} = m_1 \cap \cdots \cap m_e)$
  (2) $R \simeq R/m_1^n \times \cdots \times R/m_e^n$ where $\mathcal{N}^n = (0)$ (each R/m_j^n is Artinian & local)

- **Theorem**: $\boxed{\text{Artinian} \implies \text{dimension } 0 \text{ \& Noetherian}}$ $\quad (*)$

- Missing one part of the proof of this theorem (Lecture 26) for local Artinian rings, i.e.

Lemma 2: If (R, m) is Artinian & local (commutative), then $\dim_{R/m} m^j/m^{j+1} < \infty$. $\forall j$

Pf/ We view $k = R/m = (R/m^{j+1}) / (m/m^{j+1})$ (By 2nd Iso Thm for rings)

<u>Claim</u>: Subspaces (over k) in m^j/m^{j+1} correspond bijectively to ideals I in R/m^{j+1} with $I \subseteq m^j/m^{j+1}$ $\quad (\Rightarrow \frac{m}{m^{j+1}} I = (0))$

<u>Why?</u> $(\Rightarrow)$ $R \xrightarrow{\pi} R/m^{j+1}$ $\quad I \subset \frac{m^j}{m^{j+1}}$ k-v.sp. is also an R-module (ideal)

So $m^{j+1} \subseteq \underbrace{\pi^{-1}(I)}_{=:J} \subset \pi^{-1}(m^j/m^{j+1}) \underset{(*)}{=} m^j + m^{j+1} = m^j \Rightarrow Jm \subseteq m^{j+1}$
$\hookrightarrow m^{j+1} \subseteq m^j$

meaning $\frac{m}{m^{j+1}}$ annihilates I.

$(\Leftarrow)$ Conversely if $J \subseteq m^j/m^{j+1}$ & $\frac{m}{m^{j+1}} J = (0)$, then J is a k-module. so it is a subspace of the k-v.sp. $\frac{m^j}{m^{j+1}}$

- If $\dim_k m^j/m^{j+1} = \infty$, we can find an infinite strictly descending chain of subspaces, starting with a countable infinite, linear independent set $B = \{v_n : n \geq 1\}$ & removing one such vector at a time.

$I_1 = k\langle v_n : n \geq 1 \rangle \supsetneq I_2 := k\langle v_n : n \geq 2 \rangle \supsetneq \cdots \supsetneq I_p = k\langle v_n : n \geq p \rangle \supsetneq \cdots$

By the claim, this sequence gives an infinite strictly descending chain of ideals in R/m^{j+1} annihilated by m/m^{j+1}

This cannot happen because R/m^{j+1} is Artinian. □

TODAY we will show the converse to (*). The proof is based on the notion of "primary decomposition" (next time)

§1. Noetherian + dim 0 $\Longrightarrow$ Artinian:

Recall a key general Lemma from last time:

Lemma 1: If $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_s\}$ are pairwise coprime ideals of a commutative ring R then, so are $\{\mathfrak{a}_1^k, \ldots, \mathfrak{a}_s^k\}$ for each $k \geq 1$. Moreover, $\mathfrak{a}_1^k \cap \ldots \cap \mathfrak{a}_s^k = \mathfrak{a}_1^k \cdot \ldots \cdot \mathfrak{a}_s^k$.

Theorem 1: Let R be a commutative Noetherian ring R of dim 0 (that is, every prime ideal of R is maximal). Then, R is Artinian.

- Key fact: R Noetherian of dimension 0 $\Longrightarrow$ R has finitely many mxl ideals, say $\{m_1, \ldots, m_\ell\} = \mathrm{Max\,Spec}(R)$. ← set of mxl ideals of R.

- This will be shown by primary decomposition of (0) in R (later).
- We assume this key fact to prove Theorem 1. We mimic the proof of the Structure Thm.

Proof of Theorem 1: Since $\dim R = 0$ we have $\mathcal{N} = m_1 \cap \cdots \cap m_\ell$

- By Lemma 1 Lecture 26: $m_1 \cap \cdots \cap m_\ell = m_1 \cdots m_\ell$ (m_i's are pairwise coprime)
- R is Noetherian so $\mathcal{N}$ is a nilpotent ideal

(Pf/ Set $\mathcal{N} = \langle x_1, \ldots, x_s \rangle$ & assume $x_1^{n_1} = \cdots = x_s^{n_s} = 0$. Then, $\mathcal{N}^n = 0$ for $n = \sum_{i=1}^{s} n_i$)

- The Chinese Remainder Theorem applied to the set $\{m_1^n, \ldots, m_\ell^n\}$ of pairwise coprime ideals of R gives a surjective map

$$R \xrightarrow{\varphi} R/m_1^n \times \cdots \times R/m_\ell^n$$

with $\ker\varphi = m_1^n \cap \cdots \cap m_\ell^n \underset{\substack{\uparrow \\ \text{Lemma 1}}}{=} m_1^n \cdots m_\ell^n \underset{\substack{\uparrow \\ \text{[exercise: work with generators]}}}{\subseteq} (m_1 \cdots m_\ell)^n = \mathcal{N}^n = (0)$

So φ is a ring isomorphism.

. To finish the proof is enough to show the following 2 claims:

<u>Claim 1</u>: Finite products of Artinian rings are Artinian rings.

Pf/ Ideals of $R_1 \times \cdots \times R_\ell$ are of the form $\mathfrak{a}_1 \times \cdots \times \mathfrak{a}_\ell$ where each $\mathfrak{a}_i$ is an ideal of R_i (see HW9)

<u>Claim 2</u>: Each $R_j := R/\mathfrak{m}_j^n$ is an Artinian ring.

Pf/ By construction, each R_j is a Noetherian ring, of dimension 0 & local, with unique max'l ideal $\overline{\mathfrak{m}}_j := \mathfrak{m}_j/\mathfrak{m}_j^n$. Note: $\overline{\mathfrak{m}}_j^n = 0$.

Now, the Noetherian condition on R, says $\mathfrak{m}_j^i$ is fg as an R-module for each $i = 1, \ldots, n$. In particular $M_i := \overline{\mathfrak{m}}_j^i / \overline{\mathfrak{m}}_j^{i+1}$ is a f.g R-module, and thus a finitely generated vector space over $k := R/\mathfrak{m}_j \simeq R_j/\overline{\mathfrak{m}}_j$ ($\overline{a} \cdot \overline{x} = a \cdot \overline{x}$ for $a \in R$ & $x \in M_j$)

<u>Conclusion</u>: $\dim_k M_i < \infty$ for each $i = 1, \ldots, n$.

This condition guarantees that R_j is Artinian. Indeed, fix any descending chain $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$ of ideals of R_j. We use the same techniques used to prove (Artinian + local $\Rightarrow$ Noetherian)

For each $q = 0, \ldots, n$ we consider the chain

$$\mathfrak{a}_1 \cap \overline{\mathfrak{m}}_j^q \supseteq \mathfrak{a}_2 \cap \overline{\mathfrak{m}}_j^q \supseteq \cdots$$

Note that for $q = n$ this chain is $(0) \supseteq (0) \supseteq \cdots$

For each i, we take

$$\begin{tikzcd} \mathfrak{a}_{i+1} \cap \overline{\mathfrak{m}}_j^q \arrow[r, hook] \arrow[rr, bend right, "\varphi_{i,q}"'] & \mathfrak{a}_i \cap \overline{\mathfrak{m}}_j^q \arrow[r, two heads, "\pi"] & \dfrac{\mathfrak{a}_i \cap \overline{\mathfrak{m}}_j^q}{\mathfrak{a}_i \cap \overline{\mathfrak{m}}_j^{q+1}} \end{tikzcd}$$

. $\varphi_{i,q}$ is R_j-linear.

. $\operatorname{Ker}(\varphi_{i,q}) = \mathfrak{a}_{i+1} \cap \overline{\mathfrak{m}}_j^q \cap \mathfrak{a}_i \cap \overline{\mathfrak{m}}_j^{q+1} = \mathfrak{a}_{i+1} \cap \mathfrak{m}_j^{q+1}$.

This induces an injective R-linear map $\overline{\varphi}_{i,q} : \dfrac{\mathfrak{a}_{i+1} \cap \overline{\mathfrak{m}}_j^{\,q}}{\mathfrak{a}_{i+1} \cap \overline{\mathfrak{m}}_j^{\,q+1}} \longrightarrow \dfrac{\mathfrak{a}_i \cap \overline{\mathfrak{m}}_j^{\,q}}{\mathfrak{a}_i \cap \overline{\mathfrak{m}}_j^{\,q+1}}$

Since the map is $k = R/\overline{\mathfrak{m}}_j$ linear, we get a descending chain of k-vector spaces

$$(*) \quad \frac{\mathfrak{a}_1 \cap \overline{\mathfrak{m}}_j^{\,q}}{\mathfrak{a}_1 \cap \overline{\mathfrak{m}}_j^{\,q+1}} \xhookleftarrow{\varphi_{1,q}} \frac{\mathfrak{a}_2 \cap \overline{\mathfrak{m}}_j^{\,q}}{\mathfrak{a}_2 \cap \overline{\mathfrak{m}}_j^{\,q+1}} \xhookleftarrow{\varphi_{2,q}} \cdots$$

Since M_q is finite-dimensional, so is $\dfrac{\mathfrak{a}_1 \cap \overline{\mathfrak{m}}_j^{\,q}}{\mathfrak{a}_1 \cap \overline{\mathfrak{m}}_j^{\,q+1}}$ (it's a subspace!)

This forces the chain $(*)$ to stabilize.

Since we have finitely many chains ($0 \le q \le n-1$), we may assume that they all stabilize at the same spot (say, ℓ^{th}). This means:

$$(**) \quad \mathfrak{a}_s \cap \overline{\mathfrak{m}}_j^{\,q} \subset \mathfrak{a}_{s+1} \cap \overline{\mathfrak{m}}_j^{\,q} + \mathfrak{a}_s \cap \overline{\mathfrak{m}}_j^{\,q+1} \qquad \text{for all } s \geq \ell \text{ \& all } q = 0, \ldots, n$$

- For $q = n$, we know $\mathfrak{a}_\ell \cap \overline{\mathfrak{m}}_j^{\,n} = 0 = \mathfrak{a}_{\ell+1} \cap \overline{\mathfrak{m}}_j^{\,n} = \cdots$
- For $q = n-1$, $(**)$ yields: $\mathfrak{a}_s \cap \overline{\mathfrak{m}}_j^{\,n-1} \subseteq \mathfrak{a}_{s+1} \cap \overline{\mathfrak{m}}_j^{\,n-1} + 0$

  so $\mathfrak{a}_\ell \cap \overline{\mathfrak{m}}_j^{\,n-1} = \mathfrak{a}_{\ell+1} \cap \overline{\mathfrak{m}}_j^{\,n-1} = \cdots$
- For $q = n-2$, $(**)$ yields $\mathfrak{a}_s \cap \overline{\mathfrak{m}}_j^{\,n-2} \subseteq \mathfrak{a}_{s+1} \cap \overline{\mathfrak{m}}_j^{\,n-2} + \underbrace{\mathfrak{a}_s \cap \overline{\mathfrak{m}}_j^{\,n-1}}_{= \mathfrak{a}_{s+1} \cap \overline{\mathfrak{m}}_j^{\,n-1}}$

  So $\mathfrak{a}_s \cap \overline{\mathfrak{m}}_j^{\,n-2} \subseteq \mathfrak{a}_{s+1} \cap \overline{\mathfrak{m}}_j^{\,n-2} + \mathfrak{a}_{s+1} \cap \overline{\mathfrak{m}}_j^{\,n-1} \subseteq \mathfrak{a}_{s+1} \cap \overline{\mathfrak{m}}_j^{\,n-2}$

  Conclude: $\mathfrak{a}_\ell \cap \overline{\mathfrak{m}}_j^{\,n-2} = \mathfrak{a}_{\ell+1} \cap \overline{\mathfrak{m}}_j^{\,n-2} = \cdots$
- Continuing in this way (by reverse induction on $q \in \{0, \ldots, n\}$), we conclude: $\mathfrak{a}_\ell \cap \overline{\mathfrak{m}}_j^{\,q} = \mathfrak{a}_{\ell+1} \cap \overline{\mathfrak{m}}_j^{\,q} = \cdots$ for all $q = 0, \ldots, n$.

In particular, for $q = 0$, this gives $\mathfrak{a}_\ell = \mathfrak{a}_{\ell+1} = \cdots$ so our original chain stabilizes. $\square$

§2 Primary Ideals: Fix R = commutative ring

- Primary Decompositions are central to: (1) R Noetherian + dim 0 $\Rightarrow |\mathrm{MaxSpec}(R)| < \infty$

  (2) Structure Theorem for modules over PIDs.
- Central Tool in Algebraic Geometry & the study of Dedekind domains.

Definition An ideal $q \subsetneq R$ is **primary** if for any $a, b \in R$ we have "$ab \in q$ & $b \notin q \Rightarrow a^n \in q$ for some $n \geq 1$".

Observation: Equivalent, every zero divisor of R/q is a nilpotent element
(HW9)

Recall: The **radical** of an ideal $\mathfrak{a} \subset R$ is:

$$r(\mathfrak{a}) = \sqrt{\mathfrak{a}} = \{x \in R : x^n \in \mathfrak{a} \text{ for some } n \geq 1\}$$

Lemma: $q \subsetneq R$ primary $\Rightarrow r(q)$ is prime.

Pf/ $a, b \in \mathfrak{p} = r(q) \Rightarrow (ab)^n = a^n b^n \in q$ for some $n \geq 1$.

$$\underset{q \text{ primary}}{\Longrightarrow} \text{ either } \underset{\substack{\Downarrow \\ b \in \mathfrak{p}}}{b^n \in q} \text{ or } \underbrace{(a^n)^m \in q \text{ for some } m \geq 1}_{\substack{\Downarrow \\ a \in \mathfrak{p}}}$$

□

Observation: The difference between q & $\mathfrak{p} = r(q)$ is algebraic & highlights the difference between a fat point (pt with multiplicity) & the point viewed as a set ("Algebra" behind Algebraic Geometry).

Examples: ① $R = \mathbb{K}[x]$ (x^n) is primary & $r(x^n) = (x)$

($ab \in (x^n)$ & $x^n \nmid b \Rightarrow x \mid a$ so $a^n \in (x^n)$.)

② $R = \mathbb{K}[x,y]$ $q = (x, y^2)$ is primary but it's NOT a power of a prime ideal

Pf/ $\mathfrak{p} = r(q) = (x,y)$ and $\mathfrak{p}^2 \subsetneq q \subsetneq \mathfrak{p}$ $(x \notin \mathfrak{p}^2)$

(*) $fg \in q$ write

$$f = a_0 + x f_{1(x,y)} + y f_{2(y)} \qquad a_0 \in \mathbb{K}$$
$$g = b_0 + x g_{1(x,y)} + y g_{2(y)} \qquad b_0 \in \mathbb{K}$$

$g \notin q$ means $b_0 \neq 0$ or $(f_3(0) \neq 0$ and $b_0 = 0)$

• **Case 1**: $b_0 \neq 0$

$$\Rightarrow fg = a_0 b_0 + x(g f_1 + a_0 g_1 + g_1 y f_{2(y)}) + y(f_2 g + a_0 g_{2(y)})$$

Since $fg \in (x, y^2)$, then $a_0 b_0 = 0$, so $a_0 = 0$.

$$\Rightarrow \underbrace{fg}_{\in q} = \underbrace{x(f_1 g + g_1 f_2(y) g_1) + y^2 f_2(y) g_2(y)}_{\in q} + b_0 y f_2(y)$$

$\Rightarrow b_0 y f_2(y) \in q$ so $b_0 y f_2(y) = x h_1(x,y) + y^2 h_2(x,y)$

Evaluate at $x=0$ to get $b_0 y f_2(y) = y^2 h_2(0,y)$

$\Rightarrow y \mid f_2(y)$ so $f = 0 + x f_1(x,y) + y^2 \underbrace{\frac{f_2}{y}}_{\in R} \in q$.

- <u>Case 2</u>: $b_0 = 0$ & $y \nmid g_2(y)$

$$\Rightarrow \underbrace{fg}_{\in q} = \underbrace{x(a_0 g_1 + x f_1 g_1 + y f_2(y) g_1 + y f_1 g_2(y)) + y^2 f g_2}_{\in q} + y a_0 g_2(y)$$

$\Rightarrow y a_0 g_2(y) \in q$ so $y a_0 g_2(y) = x h_1(x,y) + y^2 h_2(x,y)$

$\overset{\Longrightarrow}{\scriptstyle x=0}$ $y a_0 g_2(y) = y^2 h_2(0,y)$ So $\underset{\text{Contra!}}{y \mid g_2(y)}$ or $\boxed{a_0 = 0}$

$\Rightarrow f = x f_1 + y f_2(y) \Rightarrow f^2 = x^2 f_1^2 + y^2 f_2^2(y) + 2xy f_1 f_2 \in (x, y^2) = q$. $\square$

<u>Note</u>: $r(q) = (x,y)$ is maximal in $\mathbb{K}[x,y]$.

This last examples are more general (see HW9)

<u>Proposition</u>: If R is commutative & $r(q)$ is maximal then q is primary.

<u>Example</u> ③ $R = \mathbb{K}[x,y,z]/(xy-z^2) \supset \mathfrak{p} = (\bar{x}, \bar{z})$

$\mathfrak{p}$ is a prime ideal but $\mathfrak{p}^2$ is not primary.

- $R/\mathfrak{p} = \mathbb{K}[x,y,z]/(x,z,xy-z^2) = \dfrac{\mathbb{K}[x,y,z]}{(x,z)} = \mathbb{K}[y]$ integral domain

- $\mathfrak{p}^2 = (\bar{x}^2, \bar{z}^2, \bar{x}\bar{z})$

$\bar{z}^2 = \bar{y}\bar{x} \in \mathfrak{p}^2$ & $\bar{x} \notin \mathfrak{p}^2$ but $\bar{y} \notin r(\mathfrak{p}^2)$. (Exercise)

⚠ We do have $\bar{y} \notin \mathfrak{p}^2$ but $\bar{x}^2 \in \mathfrak{p}^2$, i.e., the definition of primary is <u>not</u> symmetric in f & g.

Summary of examples:

- $\mathfrak{q}$ primary $\not\Rightarrow$ $\mathfrak{q}$ = power of a prime ideal
- $\mathfrak{p}$ prime $\not\Rightarrow$ $\mathfrak{p}^n$ primary.
- $r(\mathfrak{q})$ is maximal $\Longrightarrow$ $\mathfrak{q}$ primary

Next time, we'll prove the following statement ("Primary Decomposition for Noetherian rings"):

Theorem: Assume R is commutative & Noetherian & let $\mathfrak{a} \subset R$ be an ideal. Then $\exists\ \mathfrak{q}_1, \dots, \mathfrak{q}_\ell$ primary ideals of R with

$$\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_\ell$$