Lecture 28: Primary Decomposition II

Fix \( R = \text{commutative ring} \).

Recall. A primary ideal of \( R \) if \( xy \in \mathfrak{q} \ \& \ \ y \notin \mathfrak{q} \Rightarrow \ x \in \mathfrak{q} \) for non-zero \( x \).

Lemma: \( \mathfrak{q} \) primary \( \implies \) \( \mathfrak{q}(\mathfrak{q}) \) prime

Obs: 1. \( \mathfrak{q} \) primary \( \implies \) \( \mathfrak{q} \) is a power of a prime

2. \( \mathfrak{p} \) prime \( \implies \) \( \mathfrak{p}^n \) is primary

3. \( \mathfrak{q}(\mathfrak{q}) \) maximal \( \implies \) \( \mathfrak{q} \) is primary

\[ \text{§ 1. Irreducible ideals and Primary decompositions:} \]

Def: An ideal \( \mathfrak{a} \subseteq R \) is **irreducible** if \( \mathfrak{a} = b \cap c \) with \( b, c \subseteq R \) ideals, then \( \mathfrak{a} = b \) or \( \mathfrak{a} = c \).

Terminology comes from **topology**: if \( R = \mathbb{C}[x_1, \ldots, x_n] \), then \( \mathfrak{a}, b, c \) define closed sets in \( \mathbb{C}^n \) (solutions to polynomials in each ideal), namely \( V(\mathfrak{a}) \), \( V(b) \cup V(c) \). However:

\[ \mathfrak{a} = b \cap c \text{ translates to} \ V(\mathfrak{a}) = V(b) \cup V(c) \]

so we can decompose \( V(\mathfrak{a}) \), if \( \mathfrak{a} \) is not irreducible.

The next result says we have "Primary Decompositions for Noetherian rings."

**Theorem**: Assume \( R \) is **Noetherian**, Then:

(i) Every ideal in \( R \) is a finite intersection of irreducible ideals.

(ii) Irreducible \( \implies \) Primary

**Proof**: (i) We argue by contradiction. Set:

\[ \Sigma := \{ \mathfrak{a} \subseteq R \text{ ideal} : \mathfrak{a} \text{ is not a finite intersection of irreducible ideals} \} \]

Since \( \Sigma \neq \emptyset \), then \( \Sigma \) has a maximal element (R Noether), say \( \mathfrak{a} \in \Sigma \).

Since \( \mathfrak{a} \) is not irreducible (otherwise \( \mathfrak{a} \notin \Sigma \)), then \( \mathfrak{a} = b \cap c \) for a pair of ideals \( b, c \) with \( \mathfrak{a} \subseteq b \) \& \( \mathfrak{a} \subseteq c \).
Now \( t, c \notin \Sigma \) by maximality of \( \Sigma \), so:
\[
\begin{align*}
t &= t_1 \cap \ldots \cap t_k, \\
c &= c_1 \cap \ldots \cap c_l
\end{align*}
\]
with \( t_i, c_j \) irreducible.

\[
\Rightarrow \alpha = t_1 \cap \ldots \cap t_k \cap c_1 \cap \ldots \cap c_l \notin \Sigma \text{ (with } \text{ conclude: } \Sigma = \emptyset \text{.)}
\]

(ii) Fix \( \alpha \in \mathcal{R} \) irreducible ideal.

Working with \( \mathcal{R} = \mathcal{R}/\alpha \), we may assume \( (0) \) is an irreducible ideal.

Let \( x, y \in (0) \) ie. \( x = 0 \) & \( y \neq 0 \) We want to prove \( x^n = 0 \) for some \( n > 0 \).

Consider the chain of ideals:
\[
\text{Ann}(x) \leq \text{Ann}(x^2) \leq \ldots \leq \text{Ann}(x^n) \leq \ldots
\]
\[
\begin{bmatrix}
\text{Ann}(x) = \{ r \in \mathcal{R} \mid rz = 0 \} \subset \text{ideal } \mathcal{R}
\end{bmatrix}
\]

Since \( \mathcal{R} \) is Noetherian, \( \exists n > 0 \) st \( \text{Ann}(x^n) = \text{Ann}(x^{n+1}) = \ldots \).

Claim: \( (0) = (x^n) \cap (y) \).

By \( 3 \)
\[
\begin{align*}
ea \in (y) &\Rightarrow ax = 0 \quad \text{(since } xy = 0) \\
ea \in (x^n) &\Rightarrow a = bx^n
\end{align*}
\]

But \( bx^{n+1} = 0 \Rightarrow b \in \text{Ann}(x^{n+1}) = \text{Ann}(x^n) \Rightarrow bx^n = 0 \)

So \( a = bx^n = 0 \).

Since \( (0) \) is irreducible and \( (y) \neq (0) \), we conclude \( (x^n) = (0) \), i.e. \( x^n = 0 \) as desired.

**Application:**

Fix \( \mathcal{R} \) Noetherian & commutative. Let \( \alpha \notin \mathcal{R} \) be an ideal.

Write \( \alpha = q_1 \cap \ldots \cap q_l \) (a primary decomposition of \( \alpha \)).

Let \( \beta_i = (q_i) \) be the corresponding prime ideals.
Lemma 2: If \( B \not\subseteq R \) is a prime ideal, minimal among the set of prime ideals containing \( A \), then \( B = \mathfrak{p}_i \) for some \( i = 1, \ldots, n \).

By Theorem 1 of Prime Avoidance (Lecture 20), we have \( \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_l \not\subseteq B \Rightarrow \mathfrak{p}_l \not\subseteq B \) for some \( i \).

Hence \( B = \mathfrak{p}_i \) is prime.

\[ \mathfrak{p}_i \subseteq B \subseteq R \text{ minimal } \Rightarrow \mathfrak{p}_i = B. \]

Def. The minimal primes of \( R \) are the prime ideals of \( R \), minimal with respect to inclusion.

Corollary 1: There are only finitely many minimal primes over any given ideal \( \mathfrak{a} \) of a Noetherian ring \( R \). \( \iff \) min primes \( R \) are maximal.

Corollary 2: If \( R \) is Noetherian of dimension 0, the minimal primes are maximal ideals, so \( R \) has finitely many maximal ideals.

(This was the key fact assumed to prove "Noetherian + dim 0 \( \Rightarrow \) Artinian".)

§3 Reduced Primary Decompositions - Uniqueness Features:

We can simplify primary decompositions by avoiding redundancies of \( \mathfrak{q}_i \)'s & ensuring primary components have different primes associated to them (ie their radicals!).

Definition: A primary decomposition \( A = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_l \) is reduced if:

1. \( \mathfrak{q}_i = \overline{\mathfrak{q}_i} \) are all distinct
2. \( \mathfrak{q}_i \not\subseteq \mathfrak{q}_j \) for \( j = 1, \ldots, l \) (ie no \( \mathfrak{q}_i \) is redundant)

After removing redundant components (one at a time), we can achieve (1) thanks to the following lemma.
Lemma: If $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are primary ideals with $\mathfrak{p}(\mathfrak{q}_i) = \mathfrak{p}$ for $i = 1, \ldots, n$, then $\mathfrak{q} = \bigcap_{i=1}^n \mathfrak{q}_i$ is also primary if $\mathfrak{p}(\mathfrak{q}) = \mathfrak{p}$.

Proof: $\mathfrak{p}(\mathfrak{q}) = \bigcap_{i=1}^n \mathfrak{p}(\mathfrak{q}_i) = \bigcap_{i=1}^n \mathfrak{p} = \mathfrak{p}$.

We prove that $\mathfrak{q}$ is primary using the definition.

Pick $xy \in \mathfrak{q}$ with $y \notin \mathfrak{q}$. Then $\exists j$ with $y \notin \mathfrak{q}_j$.

Since $\mathfrak{q}_j$ is primary, $\exists N \geq 1$ with $x^N \in \mathfrak{q}_j$, i.e., $x \in \mathfrak{p}(\mathfrak{q}_j) = \mathfrak{p} = \mathfrak{p}(\mathfrak{q})$.

Thus $\exists N \geq 1$ with $x^N \in \mathfrak{q}$, as we wanted.

Observation: If $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ is a reduced primary decamp, the $\mathfrak{q}_i$'s are called primary components of $\mathfrak{a}$. They are not unique.

Example: $\mathfrak{a} = (x^2, xy) \subseteq \mathfrak{a} = \mathbb{K}[x, y]$ (minimal ideal)

- $\mathfrak{p}_1 = (x)$ and $\mathfrak{p}_2 = (x, y)$ are prime ideals of $\mathfrak{a}$.
- $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 = \mathfrak{p}_1 \cap (x^2, y)$ are two reduced primary decamps of $\mathfrak{a}$.
- $\mathfrak{p}_1$ is primary: $\exists g \in (x)$ and $y \notin (x) \implies x \notin \mathfrak{p}_1$, so $x \in (x)$. $(n=1)$
- $\mathfrak{p}_2$ is primary: $\exists g \in (x)$ and $y \notin (x) \implies x \notin \mathfrak{p}_2$, so $x \in (x)$. $(n=1)$
- $(x^2, y)$ is primary: $\exists g \in (x)$ and $y \notin (x) \implies x \notin (x^2, y)$.
- $\mathfrak{b}_1 = \bigcap \mathfrak{p}_i$ is a minimal prime of $\mathfrak{a} = \mathfrak{a}/\mathfrak{a}$ (it is a minimal prime of $\mathfrak{a}$).

Definition: $\text{Ass}(\mathfrak{a}) = \{ \sqrt{x} \mid \sqrt{x} \subseteq \mathfrak{a} \}$ = Associated primes.

Key results:
- $\text{Ass}(\mathfrak{a})$ does not depend on the choice of reduced primary decamp of $\mathfrak{a}$.
- Any prime containing $\mathfrak{a}$ where $\mathfrak{p} \mathfrak{a}$ is a minimal prime of $\mathfrak{a}/\mathfrak{a}$ must feature in $\text{Ass}(\mathfrak{a})$. Call them $\text{Min}(\mathfrak{a})$.
- Uniqueness of $\mathfrak{q}_i$ only applies to $\mathfrak{q}_i$'s with $\mathfrak{p}(\mathfrak{q}_i) \in \text{Min}(\mathfrak{a})$.
Theorem 2: Assume that $\mathfrak{p} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ is a reduced primary decom with $\text{Min } (\mathfrak{p}) = \{ \mathfrak{q}_1, \ldots, \mathfrak{q}_s \}$. Then $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ are uniquely determined by $\mathfrak{p}$. More explicitly: $\mathfrak{q}_i = j_i^{-1}(j_i(\mathfrak{p}) \mathfrak{R}_{\mathfrak{q}_i})$

for $i = 1, \ldots, s$ where $\mathfrak{R}_i = \sqrt{\mathfrak{q}_i}$ and $j_i : \mathfrak{R} \rightarrow \mathfrak{R}_{\mathfrak{q}_i}$.

Proof: We fix $i \in \{ 1, \ldots, k \}$ and write $\mathfrak{p} = \mathfrak{R}_i, \mathfrak{q}_i = \mathfrak{q}_i$, and $j = j_i : \mathfrak{R} \rightarrow \mathfrak{R}_{\mathfrak{q}_i}$.

Define $\mathfrak{b} = \text{ideal in } \mathfrak{R}_{\mathfrak{q}_i}$ generated by $j_i(\mathfrak{p})$

$$\mathfrak{b} = S^{-1}\mathfrak{p} = j_i(\mathfrak{p}) \mathfrak{R}_{\mathfrak{q}_i}$$

**Goal:** Show $\mathfrak{q}_i = j_i^{-1}(\mathfrak{b})$

We prove this by a series of claims:

**Claim 1:** $S^{-1}\mathfrak{p} = \bigcap_{k=1}^{s} S^{-1}(\mathfrak{q}_k)$

**Goal:** A base by induction on $l$ using that for $l = 2$, this holds for any

pair of submodules of an ambient module $M$. (Take $M = \mathfrak{R}_i, N_1 = \mathfrak{q}_i, N_2$)

More precisely.

**Proposition:** If $N_1, N_2 \subseteq M$ are $R$-submodules, then $(S^{-1}N_1) \cap (S^{-1}N_2) = S^{-1}(N_1, N_2)$

Indeed, (2) is true since $N_1 \cap N_2 \subseteq N_i \Rightarrow S^{-1}(N_1 \cap N_2) \subseteq S^{-1}(N_i)$

is $R$-submodule of $S^{-1}M$.

**Proposition:** Pick $m \in S^{-1}N_1 \cap S^{-1}N_2$, so $\exists m_1 \in N_1, m_2 \in N_2$ with $s_1, s_2 \in S$ such that $m = \frac{m_1}{s_1} = \frac{m_2}{s_2}$.

Then, $\exists t \in S$ with $t(s_2m_1 - s_1m_2) = 0$

$N_1 \ni (ts_2)m_1 = (ts_1)m_2 \in N_2$, so it's in $N_1 \cap N_2$.

Now $\frac{m}{s} = \frac{m_1}{s_1} = \frac{ts_2m_1}{ts_2s_1} \in S^{-1}(N_1 \cap N_2)$ as we wanted. \( \square \)
Claim 2: \( S^{-1}q_k = S^{-1}R \) if \( k \neq i \) so \( \tilde{S}^{-1}(S^{-1}q_k) = R \) if \( k \neq i \)

If it suffices to show \( q_k \cap S \neq \emptyset \). If this were not the case, then \( q_k \subseteq \mathfrak{p} \), then \( \mathfrak{p}_k = \mathfrak{r}(q_k) = \mathfrak{r}(q) = \mathfrak{p} \), so \( \mathfrak{a} \subseteq q_k \subseteq \mathfrak{r}(q_k) = \mathfrak{r}_k \subseteq \mathfrak{p} \) and \( \mathfrak{a} \in \min(\mathfrak{a}) \) forcing \( \mathfrak{p}_k = \mathfrak{p} \). This cannot happen because our primary decomposition was reduced!

Claim 3: \( \tilde{S}^{-1}(S^{-1}q) = q \)

Clearly \( q \subseteq \tilde{S}^{-1}(S^{-1}q) \). Note: \( q \subseteq \mathfrak{r}(q) = \mathfrak{p} \) so \( q \cap S \neq \emptyset \).

Currently, if \( x \in \tilde{S}^{-1}(S^{-1}q) \), then \( x = \sum x_i \) so

\[
\bar{x} = \sum_{i=1}^{n} \frac{a_i}{b_i} x_i \quad \text{with } a_i, b_i \in R \quad x_i \in q.
\]

Call \( a = \sum_{m=1}^{b} a_m \frac{b_m}{b_i} x_m \) to get \( \bar{x} = \frac{a}{b} \) with \( a \in q \) and \( b \notin \mathfrak{p} \).

Now: \( \exists s \in \mathfrak{p} \) with \( S(s) \bar{x} - a = 0 \) in \( R \)

\[
(s \mathfrak{b})x = sa \in q.
\]

By definition, if \( x \notin q \), then \( \exists n > 0 \) with \( (s \mathfrak{b})^n \notin q \), ie \( s \mathfrak{b} \notin \mathfrak{p} \) but this can't happen because \( s \notin \mathfrak{p} \) and \( b \notin \mathfrak{p} \).

Conclude \( x \in q \) so \( \tilde{S}^{-1}(S^{-1}q) \subseteq q \).

\( \square \)

Observation: This uniqueness will be easy to prove for PIDs, since each primary ideal \( q \neq (0) \) with \( b \) of the form \( M^n \) for some \( n \geq 1 \) and \( M \) a maximal ideal.