Lecture 28: Primary Decomposition II

Fix R = commutative ring.

Recall. $\mathfrak{q}$ primary ideal of R if $xy \in \mathfrak{q}$ & $y \notin \mathfrak{q}$ $\Rightarrow$ $x^n \in \mathfrak{q}$ for some n

Lemma 1: $\mathfrak{q}$ primary $\Longrightarrow$ $r(\mathfrak{q})$ prime

Obs: (1) $\mathfrak{q}$ primary $\not\Rightarrow$ $\mathfrak{q}$ is a power of a prime

(2) $\mathfrak{p}$ prime $\not\Rightarrow$ $\mathfrak{p}^n$ is primary

(3) $r(\mathfrak{q})$ maximal $\Rightarrow$ $\mathfrak{q}$ is primary

§1. Irreducible ideals & Primary decompositions:

Def: An ideal $\mathfrak{a} \subseteq R$ is irreducible if $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ with $\mathfrak{b}, \mathfrak{c} \subseteq R$ ideals, then $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{c}$.

- Terminology comes from topology: if $R = \mathbb{C}[x_1, \dots x_n]$, then $\mathfrak{a}, \mathfrak{b}$ & $\mathfrak{c}$ define 3 closed sets in $\mathbb{C}^n$ (solutions to polynomials in each ideal), namely $V(\mathfrak{a})$, $V(\mathfrak{b})$ & $V(\mathfrak{c})$. Moreover:

$$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \text{ translates to } V(\mathfrak{a}) = V(\mathfrak{b}) \cup V(\mathfrak{c})$$

So we can decompose $V(\mathfrak{a})$. if $\mathfrak{a}$ is NOT irreducible.

- The next result says we have "Primary Decompositions for Noetherian rings."

Theorem 1: Assume R is Noetherian. Then:

(i) Every ideal in R is a finite intersection of irreducible ideals.

(ii) Irreducible $\Rightarrow$ Primary

Proof: (i) We argue by contradiction. Set:

$$\Sigma := \{\mathfrak{a} \subseteq R \text{ ideal} : \mathfrak{a} \text{ is } \underline{\text{not}} \text{ a finite intersection of irred. ideals}\}$$

- Since $\Sigma \neq \emptyset$, then Σ has a maximal element (R Noeth), say $\mathfrak{a} \in \Sigma$

Since $\mathfrak{a}$ is not irreducible (otherwise $\mathfrak{a} \notin \Sigma$), then $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ for a pair of ideals $\mathfrak{b}, \mathfrak{c}$ with $\mathfrak{a} \subsetneq \mathfrak{b}$ & $\mathfrak{a} \subsetneq \mathfrak{c}$

Now $\mathfrak{b}, \mathfrak{c} \notin \Sigma$ by maximality of $\mathfrak{a}$, so

$$\begin{cases} \mathfrak{b} = \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_k \\ \mathfrak{c} = \mathfrak{c}_1 \cap \cdots \cap \mathfrak{c}_\ell \end{cases} \quad \text{with } \mathfrak{b}_i, \mathfrak{c}_j \text{ irreducible}$$

$\Rightarrow \mathfrak{a} = \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_k \cap \mathfrak{c}_1 \cdots \cap \mathfrak{c}_\ell \notin \Sigma$ Contr! Conclude: $\Sigma = \emptyset$.

(ii) Fix $\mathfrak{a} \subsetneq R$ irreducible ideal.

Working with $\tilde{R} = R/\mathfrak{a}$, we may assume (0) is an irreducible ideal

still Noetherian

Let $xy \in (0)$, $y \notin (0)$ ie $xy = 0$ & $y \neq 0$. We want to prove $x^n = 0$ for some $n > 0$.

Consider the chain of ideals:

$$\mathrm{Ann}(x) \subseteq \mathrm{Ann}(x^2) \subseteq \cdots \subseteq \mathrm{Ann}(x^i) \subseteq \cdots$$

$[\, \mathrm{Ann}(z) = \{ r \in R \mid rz = 0 \} \underset{\text{ideal}}{\subset} R \,]$.

Since R is Noetherian, $\exists\, n > 0$ st $\mathrm{Ann}(x^n) = \mathrm{Ann}(x^{n+1}) = \cdots$

Claim: $(0) = (x^n) \cap (y)$.

Pf/ $\supseteq$) $a \in (y) \Rightarrow ax = 0$ (since $xy = 0$)
$a \in (x^n) \Rightarrow a = bx^n$ $\Bigg\} \Rightarrow bx^{n+1} = 0$.

But $bx^{n+1} = 0 \Rightarrow b \in \mathrm{Ann}(x^{n+1}) = \mathrm{Ann}(x^n)$ so $bx^n = 0$

So $a = bx^n = 0$.

Since (0) is irreducible and $(y) \neq (0)$, we conclude $(x^n) = (0)$, ie $x^n = 0$ as desired $\square$

§2 Application:

Fix R Noetherian & commutative. Let $\mathfrak{a} \subsetneq R$ be an ideal.

Write $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_\ell$ (a primary decomposition of $\mathfrak{a}$)

irreducible ($\Rightarrow$ primary)

Let $\mathfrak{p}_i = r(\mathfrak{q}_i)$ be the corresponding prime ideals.

Lemma 2: If $\mathfrak{p} \subsetneq R$ is a prime ideal, minimal among the set of prime ideals containing $\mathfrak{a}$, then $\mathfrak{p} = \mathfrak{p}_i$ for some $i = 1, \ldots, l$

Pf/ By Theorem 2 of Prime Avoidance (Lecture 20), we have $\underbrace{q_1 \cap \cdots \cap q_l \subseteq \mathfrak{p}}_{\text{prime}} \Rightarrow q_i \subseteq \mathfrak{p}$ for some i.

Hence $\mathfrak{p}_i = r(q_i) \subseteq r(\mathfrak{p}) = \mathfrak{p}$, but

$$\mathfrak{a} \subseteq \underbrace{\mathfrak{p}_i}_{\text{prime}} \subseteq \mathfrak{p} \quad \& \ \mathfrak{p} \text{ minimal} \Rightarrow \mathfrak{p}_i = \mathfrak{p}. \qquad \square$$

Def: The minimal primes of R are the prime ideals of R, minimal with respect to inclusion.

Corollary 1: There are only finitely many minimal primes over any given ideal $\mathfrak{a}$ of a Noetherian ring R. ($\Leftrightarrow$ min primes in $R/\mathfrak{a}$)

Corollary 2: If R is Noetherian of dimension 0, the minimal primes over (0) are maximal ideals, so R has finitely many maximal ideals.

(This was the Key Fact assumed to prove "Noetherian + dim 0 $\Rightarrow$ Artinian")

§3 Reduced Primary Decompositions – Uniqueness features:

- We can simplify primary decompositions by avoiding redundancies of q_i's & ensuring primary components have different primes associated to them (ie their radicals!)

Definition: A primary decomposition $\mathfrak{a} = q_1 \cap \cdots \cap q_l$ is reduced if

(1) $\mathfrak{p}_i = \sqrt{q_i}$ are all distinct

& (2) $q_i \not\supseteq \bigcap_{j \neq i} q_j$ for $j = 1, \ldots, l$ (ie no q_i is redundant)

- After removing redundant components (one at a time), we can achieve (1) thanks to the following lemma.

Lemma: If $\tilde{q}_1, \dots, \tilde{q}_n$ are primary ideals with $r(\tilde{q}_i) = \wp$ for $i = 1, \dots, n$, then $q = \bigcap_{i=1}^{n} \tilde{q}_i$ is also primary & $r(q) = \wp$

Proof: • $r(q) \underset{[\text{exercise}]}{=} \bigcap_{i=1}^{n} r(\tilde{q}_i) = \bigcap_{i=1}^{n} \wp = \wp$.

• We prove that q is primary using the definition.

Pick $xy \in q$ with $y \notin q$. Then $\exists j$ with $y \notin \tilde{q}_j$

Since q_j is primary $\exists n \geq 1$ with $x^n \in q_j$ ie $x \in r(q_j) = \wp = r(q)$

$\Rightarrow \exists N \geq 1$ with $x^N \in q$. , as we wanted. □

Obs: If $\mathfrak{a} = q_1 \cap \dots \cap q_\ell$ is a reduces primary decomp., the q_i's are called primary components of $\mathfrak{a}$. They are not unique.

Example: $\mathfrak{a} = (x^2, xy) \subset R := \mathbb{K}[x,y]$ (monomial ideal)

- $\wp_1 = (x)$ & $\wp_2 = (x,y)$ are prime ideals of R
- $\mathfrak{a} = \wp_1 \cap \wp_2^2 = \wp_1 \cap (x^2, y)$ are 2 reduced primary decomp of $\mathfrak{a}$.
- $\wp_1$ primary: $fg \in (x)$ & $g \notin (x) \Rightarrow x \mid f$, so $f \in (x)$. $(n=1)$
- $\wp_2^2$ ——— because $r(\wp_2^2) = \wp_2$ is maximal.
- (x^2, y) ——— $r(x^2, y) = \wp_2$ ———.
- $\wp_1/\mathfrak{a} = \sqrt{\wp_1}/\mathfrak{a}$ is a minimal prime of $R/\mathfrak{a}$ ($\wp_1$ is a minimal prime over $\mathfrak{a}$)

Def: $\boxed{\mathrm{Ass}(\mathfrak{a})} = \{\sqrt{q_1}, \dots, \sqrt{q_\ell}\}$ = Associated primes.

Key results: • $\mathrm{Ass}(\mathfrak{a})$ doesn't depend on the choice of reduced primary decomp

• Any $\wp$ prime containing $\mathfrak{a}$ where $\wp/\mathfrak{a}$ is a minimal prime of $R/\mathfrak{a}$ must feature in $\mathrm{Ass}(\mathfrak{a})$. Call them $\mathrm{Min}(\mathfrak{a})$

• Uniqueness of q_i only applies to q_i's with $r(q_i) \in \mathrm{Min}(\mathfrak{a})$

Theorem 2: Assume that $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_\ell$ is a reduced primary decomp with $\mathrm{Min}(\mathfrak{a}) = \{\sqrt{\mathfrak{q}_1}, \ldots, \sqrt{\mathfrak{q}_s}\}$. Then $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ are uniquely determined by $\mathfrak{a}$. More explicitly: $\mathfrak{q}_i = j_i^{-1}(j_i(\mathfrak{a}) R_{\mathfrak{p}_i})$ for $i = 1, \ldots, s$ where $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ & $j_i : R \longrightarrow R_{\mathfrak{p}_i}$, $a \longmapsto \frac{a}{1}$.

Proof: We fix $i \in \{1, \ldots, k\}$ and write $\mathfrak{p} = \mathfrak{p}_i$, $\mathfrak{q} = \mathfrak{q}_i$ & $j = j_i : R \longrightarrow R_{\mathfrak{p}}$, $a \longmapsto \frac{a}{1}$
$S = (R \smallsetminus \mathfrak{p})$

Define $\mathfrak{b}$ = ideal in $R_{\mathfrak{p}}$ generated by $j(\mathfrak{a})$
$= S^{-1}\mathfrak{a} = j(\mathfrak{a}) R_{\mathfrak{p}}$

GOAL: Show $\mathfrak{q} = j^{-1}(\mathfrak{b})$

We prove this by a series of claims:

Claim 1: $S^{-1}\mathfrak{a} = \bigcap_{k=1}^{\ell} S^{-1}(\mathfrak{q}_k)$

Pf/ Argue by induction on ℓ using that for $\ell = 2$ (base case) this works for any pair of submodules of an ambient module M (Take $M = R$, $N_1 = \mathfrak{q}_1$, $N_2 = \mathfrak{q}_2$) More precisely.

(*) $\boxed{\text{If } N_1, N_2 \subseteq M \text{ are } R\text{-submodules, then } (S^{-1}N_1) \cap (S^{-1}N_2) = S^{-1}(N_1 \cap N_2)}$

- Indeed, $(\supseteq)$ is true since $N_1 \cap N_2 \subseteq N_i \Rightarrow S^{-1}(N_1 \cap N_2) \subseteq S^{-1}(N_i)$ are $S^{-1}R$ submodules of $S^{-1}M$.
- For $(\subseteq)$ Pick $\frac{m}{s} \in S^{-1}N_1 \cap S^{-1}N_2$, so $\exists\, m_1 \in N_1$, $m_2 \in N_2$ & $s_1, s_2 \in S$ with $\frac{m}{s} = \frac{m_1}{s_1}$ & $\frac{m}{s} = \frac{m_2}{s_2}$ in $S^{-1}M$

Then: $\exists\, t \in S$ with $t(s_2 \cdot m_1 - s_1 \cdot m_2) = 0$

$N_1 \ni (ts_2) \cdot m_1 = (t s_1) \cdot m_2 \in N_2$, so it's in $N_1 \cap N_2$

Now $\frac{m}{s} = \frac{m_1}{s_1} = \frac{ts_2 m_1}{ts_2 s_1} \in S^{-1}(N_1 \cap N_2)$ as we wanted. $\square$

Claim 2: $S^{-1}\mathfrak{q}_k = S^{-1}R$ if $k\neq i$ so $j^{-1}(S^{-1}\mathfrak{q}_k) = R$ if $k\neq i$

Pf/ It suffices to show $\mathfrak{q}_k \cap S \neq \emptyset$. If this were not the case, then $\mathfrak{q}_k \subseteq \mathfrak{p}$, then $\mathfrak{p}_k = r(\mathfrak{q}_k) \subseteq r(\mathfrak{p}) = \mathfrak{p}$, so

$\mathfrak{a} \subseteq \mathfrak{q}_k \subseteq r(\mathfrak{q}_k) = \mathfrak{p}_k \subseteq \mathfrak{p}$ & $\mathfrak{p}\in \mathrm{Min}(\mathfrak{a})$ forcing $\mathfrak{p}_k = \mathfrak{p}$

This cannot happen because our primary decomposition was reduced!

Claim 3: $j^{-1}(S^{-1}\mathfrak{q}) = \mathfrak{q}$

Pf/ Clearly $\mathfrak{q} \subseteq j^{-1}(S^{-1}\mathfrak{q})$ Note: $\mathfrak{q} \subseteq r(\mathfrak{q}) = \mathfrak{p}$ so $\mathfrak{q}\cap S = \emptyset$

Conversely, if $x \in j^{-1}(S^{-1}\mathfrak{q})$, then $x = \frac{x}{1} \in S^{-1}\mathfrak{q}$, so

$$\frac{x}{1} = \sum_{m=1}^{N} \frac{a_m}{b_m}\frac{x_m}{1} \quad \text{with } \frac{a_m}{b_m} \in R_{\mathfrak{p}},\ x_m \in \mathfrak{q}.$$

$$= \frac{\sum_{m=1}^{N} a_m \frac{b}{b_m} x_m}{b} \quad \text{with } b = \prod b_m \notin \mathfrak{p}$$

Call $a = \sum_{m=1}^{N} a_m \underbrace{\frac{b}{b_m}}_{\in R} x_m \in \mathfrak{q}$, to see $\frac{x}{1} = \frac{a}{b}$ with $a \in \mathfrak{q}$ & $b\notin\mathfrak{p}$

Now: $\exists\, s \notin \mathfrak{p}$ with $s(bx - 1\cdot a) = s(bx-a) = 0$ in R

$(sb)x = sa \in \mathfrak{q}$.

By definition, if $x \notin \mathfrak{q}$, then $\exists\, n>0$ with $(sb)^n \in \mathfrak{q}$, ie $sb \in \mathfrak{p} = r(\mathfrak{q})$

but this can't happen because $s\notin\mathfrak{p}$ & $b\notin\mathfrak{p}$.

Conclude $x\in\mathfrak{q}$. so $j^{-1}(S^{-1}\mathfrak{q}) \subseteq \mathfrak{q}$. $\square$

Observation: This uniqueness will be easy to prove for PIDs, since each primary ideal $\mathfrak{q}\neq(0)$ will be of the form $\mathfrak{m}^n$ for some $n\geq 1$ & with $\mathfrak{m}$ a maximal ideal.