Lecture 30: Modules over PIDs, Free / Torsion Modules

Recall: PIDs = commutative domain + ideals admit 1 generator.

§1. Modules over PIDs

GOAL for next lectures: Classify finitely generated modules over PID's

Applications: ① Finitely generated abelian gps = fg $\mathbb{Z}$-modules

$\rightsquigarrow$ Classification of f.g ab. gps.

② $\mathbb{K}[x]$-modules = $\mathbb{K}$ Vector-spaces M + a fixed $\mathbb{K}$-linear map $f: M \to M$ ($f = \cdot x$)

$\rightsquigarrow$ Jordan forms for matrices in $\mathrm{Mat}_{n\times n}(\mathbb{C})$, Smith normal forms in $\mathrm{Mat}_{n\times n}(\mathbb{Z})$ or $\mathrm{Mat}_{n\times n}(R)$ if R = PID

Overview:

- Elements of a module M come in 2 flavours:
  - $\mathrm{Ann}(m) = (0)$ $\rightsquigarrow$ m is a "free" element
  - $\mathrm{Ann}(m) \neq (0)$ $\rightsquigarrow$ $\mathrm{Ann}(m) = (f)$ $f \neq 0$

    $\cap\!|$ R ideal. So m is a "torsion element" $\rightsquigarrow m \in M_{tor}$

$\rightsquigarrow$ M will be decomposed into a "free part" and a "torsion part".

- Free part $\simeq \bigoplus_{\alpha \in I} R = R^{\oplus I}$ ($|I| < \infty$ if M is f.g.)
- Torsion part: $M_{tor} = N$ will be decomposed unique in 2 canonical ways

Method 1: $N \simeq \bigoplus_{\substack{p_i \text{ prime} \\ p_i \in R}} N_{p_i^{n_i}}$ where $N_a := \mathrm{Ker}\,(N \xrightarrow{a} N)$ for $a \in R$ ($x \longmapsto a\cdot x$)

with $N_{p_i^{n_i}} \simeq R/(p_i^{a_1^{(i)}}) \oplus \cdots \oplus R/p_i^{a_s^{(i)}}$ and $n_i = a_1^{(i)} \geq \cdots \geq a_s^{(i)} \in \mathbb{Z}_{\geq 0}$ (uniquely determined by N & p_i)

Method 2: $N \simeq R/(q_1) \oplus \cdots \oplus R/(q_r)$ with $q_i \neq 0$, q_i not units of R

& $q_r | q_{r-1} | \cdots | q_1$ (uniqueness = ideals $(q_1), \ldots, (q_r)$ are unique).

Key: Primary decompositions for PIDs.

TODAY: First part of classification of modules over PIDs (torsion / free parts)

We work with any commutative ring R whenever possible.

§2. Free modules:

Def: An R-module M is *free* if $M \underset{\varphi}{\xleftarrow{\simeq}} \bigoplus_{i\in I} R$ $(= R^{\oplus I})$ for some I.

We say $\{\varphi(e_i) : i \in I\}$ is a basis for M.

Theorem 1: If R is commutative an M is a free module, then any two bases for M have the same cardinality. (Name = rank(M))

Pf/ Consider $\mathfrak{m}$ a maximal ideal on R. Then $\overline{M} = M/\mathfrak{m}M$ is a k-vector space for $k = R/\mathfrak{m}$ (field). Then, $\overline{M}$ has a basis & all bases of $\overline{M}$ have the same cardinality.

- Furthermore: If $(x_i)_{i\in I}$ is a basis for M, then $\overline{x_i} = x_i + \mathfrak{m}M$ is a basis for $\overline{M}$ (so $|I| = \dim_k \overline{M}$ does not depend on the basis)

Indeed, we show $\{\overline{x_i}\}_{i\in I}$ both spans & is li:

① $\{\overline{x_i}\}_{i\in I}$ spans: If $\overline{x} \in M/\mathfrak{m}M$, then

$$x = \sum_{\substack{i\in I\\ \text{finite}}} a_i x_i \quad \text{with } a_i \in R \quad \text{so} \quad \overline{x} = \sum_{\substack{i\in I\\ \text{finite}}} (a_i + \mathfrak{m})\, \overline{x_i}$$

② $\{\overline{x_i}\}_{i\in I}$ is li: If $\overline{0} = \sum_{\substack{i\in I\\ \text{finite}}} (a_i + \mathfrak{m})\, \overline{x_i}$, then

$$\sum_{\substack{i\in I\\ \text{finite}}} a_i x_i \in \mathfrak{m}M, \text{ so } x = \sum_{\substack{i\in I\\ \text{finite}}} a_i x_i = \sum_{j=1}^{m} \overset{\in\mathfrak{m}}{b_j}\, \overset{\in M}{y_j} \quad \text{With } y_j := \sum_{\substack{i\in I\\ \text{finite}}} c_i^{(j)} x_i$$

$$\rightsquigarrow \sum_{j=1}^{m} b_j \sum_{\substack{i\in I\\ \text{finite}}} c_i^{(j)} x_i \underset{\text{regroup}}{=} \sum_{\substack{i\in I\\ \text{finite}}} \underbrace{\left(\sum_{j=1}^{m} b_j c_i^{(j)}\right)}_{\in\mathfrak{m}} x_i .$$

By definition of $\bigoplus_{i\in I} R$, $\begin{cases} a_i \in \mathfrak{m} & \forall i \text{ in supp of } X. \\ a_i = 0 & \text{otherwise}. \end{cases}$ Thus: $a_i + \mathfrak{m} = 0 \ \forall i$

& We conclude $\{\bar{x}_i\}$ is l.i. $\square$

• Next, we need to ensure freeness is preserved for submodules:

This is not true for general commutative rings!

<u>Example</u>: $R = K[x,y]$ $\quad M = R$ is free of rank 1. but

$I = (x,y)$ is not a free submodule

• $I \not\simeq R$ (not a cyclic module)

• We have the obvious relation: $y\underline{\underline{x}} - x\underline{\underline{y}} = 0$

in I

• Any $\{f_i\}_{i\in I}$ generating set will have obvious rel'n $f_i \underbrace{f_j}_{} - f_j \underbrace{f_i}_{} = 0$. $\square$

in I

<u>Theorem 2</u>: Let F be a free module over a PID R & M be a submodule.

Then, M is free and $\operatorname{rank}(M) \leq \operatorname{rank}(F)$.

<u>Proof</u>: We discuss the finite case (For the case when $\operatorname{rank}(F)$ is infinite, see HW10). Assume F has a basis $\{x_i\}_{i=1}^n$ ($n = \operatorname{rank} F$)

Let $M_r = M \cap (x_1, \dots, x_r)$ for $r = 1, \dots, n$.

We show M_r is free of rank $\leq r$ by induction on r:

• <u>Base case</u>: $r = 1$ $\quad M_1 = M \cap (x_1)$ is a submodule of (x_1), so

$M_1 = (a_1 x_1)$ for some $a_1 \in R$.

So $M_1 = 0$ or free with $M_1 \simeq R$ because $\operatorname{Ann}(x_1) = 0$

& R is a domain.

- Inductive step: Consider the following set of R

$$\mathfrak{a} = \{a \in R : \exists\ x \in M \text{ with } x = b_1 x_1 + \cdots + b_r x_r + a x_{r+1} \text{ for } b_1, \ldots, b_r \in R\}$$

Claim: $\mathfrak{a}$ is an ideal (because M is an R-module)

Since R is a PID, then $\mathfrak{a} = (a_{r+1})$ for some $a_{r+1} \in R$.

- We treat two cases:

(1) If $a_{r+1} = 0$, then $M_{r+1} = M_r$ so M_{r+1} is free of rank $\leq r$.

(2) If $a_{r+1} \neq 0$, we pick $w \in M_{r+1}$ with $w = \underbrace{b_1 x_1 + \cdots + b_r x_r}_{\in (x_1, \ldots, x_r)} + a_{r+1} x_{r+1}$

For any $x \in M_{r+1}$ we write $x = a_1 x_1 + \cdots + a_r x_r + \underbrace{(c\, a_{r+1})}_{\in \mathfrak{a}} x_{r+1}$

so $x - cw \in M \cap (x_1, \ldots, x_r) = M_r$

So $\boxed{M_{r+1} = M_r + (w)}$

Clearly, $M_r \cap (w) = (0)$

$$\left.\right\} \quad M_{r+1} = \underset{\substack{\cong \\ R^s \\ s \leq r}}{M_r} \oplus \underset{\substack{\cong (*) \\ R}}{(w)}$$

(*) $\mathrm{Ann}(w) = (0)$ because $a_{r+1} \neq 0$. & $\{x_1, \ldots, x_s\}$ is l.i. □

Corollary: If E is a f.g module over a PID (R) & E' is a submodule then E' is f.gen.

Pf/ View $E = R^n/\text{rels}$. $\varphi: R^n \twoheadrightarrow R^n/\text{rels} = E$

Then $M = \varphi(E') \subseteq R^n$ so free of rank $\leq n$. If $B = \{x_1, \ldots x_s\}$ is a basis for M, then $E' = \varphi(M) = \langle \varphi(x_1), \ldots \varphi(x_s) \rangle$ is a fin. gen. R-module. □

Alternative Proof: Use PID $\Rightarrow$ Noetherian

- E f.g module & R Noeth, so E is Noetherian as an R-module & $E' \subseteq E$ is f.g as a submodule. □

§3. Torsion for modules:

Def: Let M be an R-module. We say M is a __torsion module__ if given $x \in M$ $\exists\, a \in R \setminus \{0\}$ with $ax = 0$ (equivalently, $\text{Ann}(x) \neq (0)$ $\forall x \in M$).

Obs: Finite abelian gp translates to finitely generated torsion module over $\mathbb{Z}$.

Def: A __torsion element__ x of a module M is an element with $\text{Ann}(x) \neq (0)$. Write $\boxed{M_{tor}} = \{\text{torsion elements of } M\}$

Def If $M_{tor} = \{0\}$, we say M is __torsion free__.

⚠ Torsion free + fg $\not\Longrightarrow$ Free for modules over a general ring R

Ex: $M = (x, y)$ torsion free $\mathbb{K}[x,y]$-mod, but NOT free.

However, the statement is true for modules over PIDs:

Proposition 1: M a fg module over a PID $= R$. If M is torsion free, then M is free.

Proof Next Time

This proposition will allow us to show:

Theorem 3: Fix R a PID and M a f.g R-module. Then, M/M_{tor} is a free R-module. Furthermore, there exists a free submodule F of M with $M = M_{tor} \oplus F$. The rank of F is uniquely determined by M & $F \simeq M/M_{tor}$ (as R-modules)