Lecture 31: Modules over PIDs II - Classification

Recall: · Last time we talked about free modules over a PID R

$$M \cong \bigoplus_{i\in I} R \quad \text{via a basis } \{e_i\}_{i\in I} \quad (\text{generates} + LI/R)$$

· Defined <u>Torsion elements</u>: $x \in M$ with $\mathrm{Ann}(x) \neq (0)$

$M_{Tor} = \{$ torsion elements $\}$ submodule of M

· M is <u>Torsion free</u> module $\Leftrightarrow$ $M_{Tor} = \{0\}$

Note: M_{Tor} is fg if M is because R is Noetherian (M fg $\Rightarrow M$ is a Noetherian module)

§1. Torsion and free parts:

Proposition 1: Fix a fg module $M \neq (0)$ over a PID $= R$. If M is torsion free, then M is free.

Pf/ Consider $S = \{v_1, \ldots, v_n\}$ a <u>maximal li</u> set of elements of M among a set $Y = \{y_1, \ldots, y_m\}$ of generators of M.

(Here: <u>li</u> means $a_1 v_1 + \cdots + a_n v_n = 0 \quad a_i \in R \Rightarrow a_i = 0 \ \forall i$)

• Pick $y \in Y \setminus S$. Then: $\exists\, a, b_1, \ldots, b_n \in R$, not all 0, so

$$ay + b_1 v_1 + \cdots + b_n v_n = 0.$$

Since S is li, then $a \neq 0$.

• If $y \in S$, then $y = v_i$ & $\overset{a}{1} \cdot y - 1 \cdot v_i = 0$.

<u>Conclusion</u>: For all $j = 1, \ldots, m$ we can find $a_j \in R \setminus \{0\}$ with $a_j y_j \in (v_1, \ldots, v_n)$. Take $a = a_1 \cdots a_m$ (need f.g of M for this!)

Then, $aM \subseteq (v_1 \ldots v_n)$ & $a \neq 0$ (R domain)

We take the multiplication map $\varphi_a : M \longrightarrow (v_1, \ldots v_n) =: N$ is free!

$$m \longmapsto am.$$

• φ_a is injective because M is torsion free.

• $N = (v_1, \ldots, v_n)$ is a free module ($\simeq R^n$), so $\varphi_a(M) \subset N$ is free (Theorem 2 Lecture 30)

Conclude: $M \simeq \varphi_a(M)$ is free of rank $\leq$ rank $N = n \leq m = \#$ gens of M

$\Rightarrow$ rank $M \leq \min\{\#(\text{gen set}) \text{ for } M\}$. $\square$

The previous proposition allows us to decompose fg modules over PID's as a direct sum of a torsion & a free-module.

Theorem 1: Fix R a PID and M a f.g R-module. Then, M/M_{Tor} is a free R-module. Furthermore, there exists a free submodule F of M with $M = M_{Tor} \oplus F$. The rank of F is uniquely determined by M & $F \simeq M/M_{Tor}$.

Proof: • We first show that $\overline{M} = M/M_{Tor}$ is torsion free.

Let $\overline{x} \in \overline{M}$ and $b \in R$ with $b\overline{x} = 0$ in $\overline{M}$. Then, $bx \in M_{Tor}$ so $\mathrm{Ann}(bx) \neq (0)$

But $\mathrm{Ann}(bx) = (c)$ $c \neq 0$ gives $(bc)x = 0$ in M.

So either $bc = 0$ or $x \in M_{Tor}$ $(\Rightarrow \overline{x} = 0)$

$\Downarrow c \neq 0$

$b = 0$. & $\overline{x}$ is not a torsion element of $\overline{M}$.

• M is fg, so $\overline{M}$ is fg

• $\overline{M}$ is fg & torsion free. By Proposition 1, it is a free R-module. It's rank is uniquely determined by M.

• To find F, we need a lemma (applied to $\varphi: M \twoheadrightarrow M/M_{Tor}$)

Lemma: Consider M & M' two modules over a PID R. Assume M' is free & let $f: M \twoheadrightarrow M'$ be a surjective R-linear map. Then, there exists a free submodule N of M such that

(1) $f|_N$ induces an isomorphism $f|_N : N \xrightarrow{\sim} M'$. of R modules

(2) $M = N \oplus \mathrm{Ker} f$.

(Analog of Rank-Nullity theorem for modules over PIDs)

Proof: Pick a basis $B = \{x_i'\}_{i \in I}$ for M'. For each i, let $x_i \in M$ with $f(x_i) = x_i'$.

Take $N = (x_i : i \in I)$

Claim: $\{x_i : i \in I\}$ is li

Pf/ $\sum_{\substack{i\in I\\ \text{finite}}} a_i x_i = 0 \rightsquigarrow \sum_{\substack{i\in I\\ \text{finite}}} a_i \underbrace{f(x_i)}_{=x_i'} = 0 \underset{\mathcal{B}\text{ basis}}{\Longrightarrow} a_i = 0 \ \forall i.$

Conclusion: N is free with basis $\{x_i\}_{i\in I}$.

- Clearly: $f|_N : N \twoheadrightarrow M'$
- For $x \in M$, we can find $a_i \in R$ (finitely many $\neq 0$) with

$$f(x) = \sum_{\substack{i\in I\\ \text{finite}}} a_i x_i' = \sum_{\substack{i\in I\\ \text{finite}}} a_i f(x_i)$$

so $x - \sum_{\substack{i\in I\\ \text{finite}}} a_i x_i \in \ker f$. Thus, $M = N + \ker f$

- $N \cap \ker f = (0)$ because $\{x_i'\}_{i\in I}$ is a basis.

So $f|_N : N \xrightarrow{\sim} M'$ & $M = N \oplus \ker f$. □

§2 Classification v.1:

Def: An element $p \in R\setminus\{0\}$ is prime if the ideal (p) is prime

- From now on, we fix a list of prime element representatives $\{p_i\}_{i\in\Sigma}$ in R (meaning $(p_i) \neq (p_j)$ $\forall i \neq j$.

Recall $M_a := \ker(M \xrightarrow{a\cdot} M)$ for all $a \in R$
$m \longmapsto a\cdot m$

- **Def** A fg R-module M is a p-torsion module if $M = M_{p^n}$ for some $n \geq 1$.

Classification Theorem 1: If $M \neq (0)$ is a fg torsion module over a PID R, then:

$$M = \bigoplus_{\substack{p_i \text{ prime}\\ \text{finite}}} M_{p_i^{n_i}}$$ for suitable $n_i \in \mathbb{Z}_{\geq 0}$ with $M_{p_i^{n_i}} \neq \{0\}$. (finitely many!)

Furthermore: $M_{p_i^{n_i}} \cong R/(p_i^{v_1^{(i)}}) \oplus \cdots \oplus R/(p_i^{v_s^{(i)}})$ with $n_i = v_1^{(i)} \geq v_2^{(i)} \geq \cdots \geq v_s^{(i)} \geq 1$.

The sequence $(v_j^{(i)})$ is uniquely determined by M & p_i.

Lecture 29: Write $\mathrm{Ann}(M) = (a)$ $a \neq 0$ $a \notin R^\times$ as:

$a = u\, p_{i_1}^{n_{i_1}} \cdots p_{i_r}^{n_{i_r}}$ $\rightsquigarrow$ $M = M_a = \bigoplus_{j=1}^{r} M_{p_{i_j}^{n_{i_j}}}$
$u \in R^\times$ p_{i_j} prime reps $n_{i_j} \geq 1$ $\hookrightarrow$ Exercise (induct on r)

- Uniqueness of p_{ij} & n_{ij} follows from uniqueness of primary decomposition for PIDs.

NEXT: We focus on the classification of p-torsion modules. Uniqueness will be proven at the end.

- Assume $M = M_{p^n}$ with n minimal. Notice $\overline{M} = M/pM$ is a k-vsp with $k = R/(p)$ (in a PID prime & $\neq 0$ $\Rightarrow$ mxl)

- Since M is fg: $\dim_k \overline{M} < \infty$

- We argue by induction on $\dim_k \overline{M}$, using the following lemma for the inductive step.

Lemma: Assume $\mathrm{Ann}(M) = (p^n)$ & pick $x \in M$ with $\mathrm{Ann}(x) = (p^n)$

Consider the ses $0 \longrightarrow (x) \longrightarrow M \xrightarrow{\pi} N \longrightarrow 0$, $N = M/(x)$

Then (1) $\dim_k N/pN < \dim_k M/pM$

(2) If N decomposes as in the Classification Thm, then π admits a section. (use (1) + IH)

Pf/ (1)

$$\begin{tikzcd} M \arrow[r, two heads, "\pi_1"] \arrow[d] \arrow[rr, bend right, "f"'] & N \arrow[r, two heads, "\pi_2"] & N/pN \\ M/pM \arrow[urr, "\overline{f}"'] & & \end{tikzcd}$$

$\pi_1(pM) = p\pi_1(M) = pN$

so $pM \subseteq \ker f$

we get $\overline{f} : \dfrac{M}{pM} \twoheadrightarrow N/pN$

- $\overline{f}$ is k-linear map
- $\overline{f}$ is surjective so $\dim_k N/pN \leq \dim_k \dfrac{M}{pM} < \infty$

- $(x) \in M$ satisfies $\overline{f}(x + pM) = 0$. & $x + pM \neq pM$ because $p^{n-1} x \neq 0$. So $\ker \overline{f} \neq (0)$

The Rank-Nullity Theorem for k-linear maps on finite dimensional vector spaces implies:

$$\dim_k \frac{N}{pN} < \dim_k \frac{M}{pM}. \qquad \text{(this proves (1)).}$$

(2) We assume $N \overset{\varphi}{\cong} \bigoplus_{i=1}^{s} R/(p^{\nu_i})$ with $\nu_1 \geq \cdots \geq \nu_s$

Consider $\{\bar{y}_1, \ldots, \bar{y}_s\}$ where $\varphi(y_i) = e_i$. $\mathrm{Ann}(\bar{y}_i) = p^{\nu_i}$.

We want to lift each y_i to M so that $\begin{cases} \cdot\ \mathrm{Ann}(y_i) = \mathrm{Ann}(\bar{y}_i) \\ \cdot\ y_i + R(x) = \bar{y}_i \in N \end{cases}$

(This gives the section $N \ni \sum_{i=1}^{s} a_i \bar{y}_i \longmapsto \sum a_i y_i \in M$)

- It suffices to do this for a single $\bar{y} \in N \smallsetminus \{0\}$

Assume $\mathrm{Ann}(\bar{y}) = (p^{l})$ for some $l \geq 1$. Pick an $y \in M$ with $y + R(x) = \bar{y}$.

Then $p^l y \in R(x)$. Write $p^l y = bx$ for $b \in R$ & factor b as $b = p^s c$ with $p \nmid c$ & $s \geq 0$.

- Since $p^n x = 0$ we may assume $\boxed{s \leq n}$ (otherwise, $p^s c x = 0 = p^n c x$) so we replace b by $p^n c$.
- If $s = n$, then $p^l y = 0$, $p^{l-1} y \notin R(x)$ so $p^{l-1} y \neq 0$ $\Big\}$ $\mathrm{Ann}(y) = \mathrm{Ann}(\bar{y})$ (so y works!)
- If $s < n$, then $\mathrm{Ann}(p^s c x) = (p^{n-s})$ so $\mathrm{Ann}(y) = (p^{l+n-s}) \supseteq \mathrm{Ann}(M) = (p^n)$

Since $p^n y = 0$ we get $l + n - s \leq n$, i.e. $\boxed{l \leq s}$

So $\boxed{y' = y - p^{s-l} c x}$ satisfies

- $y' + R(x) = \bar{y}$ & $\mathrm{Ann}(y') = (p^l) = \mathrm{Ann}(\bar{y})$.

$(p^l y' = p^l y - p^s c x = 0$ & $p^{l-1} y' = p^{l-1} y - p^{s-1} c x = 0$ forces $p^{l-1} y \in R(x)$ Contr.!)

- Assume we've lifted $\bar{y}_1, \ldots, \bar{y}_s$ to $y_1, \ldots, y_s$ with $\mathrm{Ann}(y_i) = \mathrm{Ann}(\bar{y}_i)$ & $y_i + R(x) = \bar{y}_i \in N$

Then $M = R(x) \oplus M'$ where $M' = R(y_1 \cdots y_s)$

Since
- $M' \cap R(x) = \{0\}$
- $M/R(x) = M' \simeq N$ so π has a section! $N \longrightarrow M'$ □

<u>End of the proof of Classification Thm.</u> (existence of the decomposition)

We proceed by induction on $\dim_k M/pM$ ($k = R/(p)$ field)

- M fg so $\dim_k M/pM < \infty$. Here, $M = M_{p^n}$ with n minimal $n \geq 1$

<u>Base case:</u> $\dim_k M/pM = 1 = \dim_k (x)/p(x)$ forces $M = (x)$ because $\dim_k N/pN = 0$ so $N = pN = p^2N = \cdots = p^nN \overset{N \subseteq M \,\&\, \mathrm{Ann}(M) = (p^n)}{=} 0$, ie $N = M/(x) = 0$. Thus, $M \simeq R/(p^n)$.

<u>Inductive step:</u> Assume N admits a decomp. since $\dim_k N/pN < \dim_k M/pM$.

Using the section to $M \longrightarrow N$ from the proof of the lemma we build $y_1, \ldots, y_s$ from the decomp of N, ie $N = \langle \bar{y}_1 \cdots \bar{y}_s \rangle \longrightarrow M' = \langle y_1 \cdots y_s \rangle$ with $\mathrm{Ann}(y_i) = \mathrm{Ann}(\bar{y}_i)$

- Since $N = \bigoplus_{i=1}^{s} R(\bar{y}_i) \simeq \bigoplus_{i=1}^{s} R/(p^{\nu_i})R$ with $\mathrm{Ann}(\bar{y}_i) = (p^{\nu_i})$, $\nu_1 \geq \cdots \geq \nu_s$

and $(p^n) = \mathrm{Ann}(M) \subseteq \mathrm{Ann}(M') = (p^{\nu_1})$, we get $\boxed{n \geq \nu_1 \geq \cdots \geq \nu_s}$.

- To finish, we show $\boxed{R(y_1 \cdots y_s) = R(y_1) \oplus \cdots \oplus R(y_s)}$

Indeed if $a_1y_1 + \cdots + a_sy_s = 0$, we want to show $a_iy_i = 0 \ \forall i$

Viewed in N: $a_1\bar{y}_1 + \cdots + a_s\bar{y}_s = 0$ in $N = \bigoplus_{i=1}^{s} R(\bar{y}_i)$ forces

$a_i\bar{y}_i = 0$ so $a_i \in \mathrm{Ann}(\bar{y}_i) = \mathrm{Ann}(y_i) \Rightarrow a_iy_i = 0$

Thus $M = \underset{\underset{R/(p^n)}{\wr\wr}}{(x)} \oplus \underset{\underset{R/(p^{\nu_1})}{\wr\wr}}{R(y_1)} \oplus \cdots \oplus \underset{\underset{R/(p^{\nu_s})}{\wr\wr}}{R(y_s)}$ $\quad n \geq \nu_1 \geq \cdots \geq \nu_s$ □

This construction ensures the existence of the decomposition.

Uniqueness Proof: $M_{p^n} = \bigoplus_{i=1}^{s} R(x_i) \simeq \bigoplus_{i=1}^{s} R/(p^{\nu_i})$ $\quad n \geq \nu_1 \geq \cdots \geq \nu_s$

We need to show both s & the sequence $\nu_1 \geq \cdots \geq \nu_s$ is unique.

Note $\mathrm{Ann}(M_{p^n}) = (p^n) = \mathrm{Ann}\left(\bigoplus_{i=1}^{s} R/(p^{\nu_i})\right) = (p^{\nu_1})$ forces $\boxed{\nu_1 = n}$

$\nu_1 \geq \cdots \geq \nu_s$

Next, we determine s:

Claim: $s = \dim_k M/pM$ $\quad (k = R/(p)$ field$)$

Pf/ Since: $\quad M = R(x_1) \oplus R(x_2) \oplus \cdots \oplus R(x_s)$

Then $\quad pM = pR(x_1) \oplus pR(x_2) \oplus \cdots \oplus pR(x_s)$

with $\nu_{t+1} = \cdots = \nu_s = 1$. (could have $t = s$)

So $\dfrac{M}{pM} \simeq \dfrac{R(x_1)}{pR(x_1)} \oplus \dfrac{R(x_2)}{pR(x_2)} \oplus \cdots \oplus \dfrac{R(x_t)}{pR(x_t)} \oplus \bigoplus_{j=t+1}^{s} R(x_s)$

k v.s. $\quad \simeq k \quad \simeq k \quad \simeq k \quad \simeq k$

$\dim_k M/pM = t + (s-t) = s$

- So # of Terms is unique!

- Assume $M = \bigoplus_{i=1}^{s} R(x_i')$ with $\mathrm{Ann}(x_i') = \nu_i'$ & $\nu_1' \geq \nu_2' \geq \cdots \geq \nu_s'$

want to show $\nu_i' = \nu_i$ for all $i = 1, \ldots, s$

But $\mathrm{Ann}(M) = (p^{\nu_1'}) = (p^n)$ yields $\nu_1' = n = \nu_1$

- Next, we consider the multiplication map $p: M \xrightarrow{p\cdot} M$ & we argue by induction on n where $\mathrm{Ann}(M) = (p^n)$.

$\tilde{M} = pM$ has $\mathrm{Ann}(\tilde{M}) = (p^{n-1})$

- If $n=1$, then $pM = 0$ so $\nu_1 = \nu_2 = \cdots = \nu_s = 1$

$\nu_1' = \nu_2' = \cdots = \nu_s' = 1$

• Assume $n \geq 2$ & that the decomp is unique for any module with $\mathrm{Ann}(M) = (p^k)$ for $k \leq n-1$. By construction the decomp of M' is unique.

On the other hand, $M' = pM = \bigoplus_{i=1}^{r} R(px_i)$ $\quad$ $\nu_r \geq 2$ & $\nu_{r+1} = \cdots = \nu_s = 1$

$= \bigoplus_{i=1}^{r'} R(px_i')$ $\quad$ $\nu_{r'} \geq 2$ & $\nu'_{r'+1} = \cdots = \nu'_s = 1$

and $\mathrm{Ann}(px_i) = (p^{\nu_i - 1})$ for $i \leq r$

$\mathrm{Ann}(px_i') = (p^{\nu_i' - 1})$ for $i \leq r'$

Our inductive hypothesis gives: $r = r'$ &

$\nu_i - 1 = \nu_i' - 1 \quad \forall i = 1, \ldots, r$

& $\nu'_{r+1} = \cdots = \nu'_s = 1 = \nu_s = \cdots = \nu_{r+1}$ $\quad$ This concludes our proof. $\square$

Observation:

For M_{p^n} with $\mathrm{Ann}(M_{p^n}) = (p^n)$: $n =$ exponent of $\mathrm{Ann}(M_{p^n}) = (p^n)$ $\quad \mathrm{Ann}(x) = (p^n)$

$\nu_1 =$ ———————— $(M_{p^n}/(x)) = (p^{\nu_1})$

$\nu_2 =$ ———————— $(M_{p^n}/(x, y_1)) = (p^{\nu_2})$

$\rightsquigarrow$ This is how we compute $\nu_1, \nu_2, \ldots$

§2 Classification v2:

We rearrange the $M_{p_i^{\nu_j^{(i)}}}$ factors to give an alternative classification:

Classification Thm v2: If $0 \neq M$ is a f.g. torsion module over a PID R, then $M \simeq R/(q_1) \oplus \cdots \oplus R/(q_r)$

where $q_i \neq 0$, $q_i \notin R^\times$ $\forall i$ & $q_r \mid q_{r-1} \mid \cdots \mid q_1$.

Furthermore, the sequence of ideals $(q_1), \ldots, (q_r)$ is uniquely determined by the above conditions.

Pf/ Write $M = M_{p_1^{n_1}} \oplus \cdots \oplus M_{p_r^{n_r}}$ with

$$M_{p_i^{n_i}} \cong R/(p_i^{\nu_1^{(i)}}) \oplus \cdots \oplus R/(p_i^{\nu_{s_i}^{(i)}}) \quad \text{with } \nu_1^{(i)} \geq \cdots \geq \nu_{s_i}^{(i)} \geq 1$$

- We complete with $\nu_j^{(i)} = 0$ $\forall j > s_i$ so that all decomp have the same number of summands. $s = \max_{1 \leq i \leq r} \{s_i\}$
- We regroup by columns:

$$\begin{array}{ccccccc} M_{p_1^{n_1}} & = & \boxed{R/(p_1^{\nu_1^{(1)}})} & \oplus & \cdots\cdots & \oplus & \boxed{R/(p_1^{\nu_s^{(1)}})} \\ \oplus & & \oplus & & & & \oplus \\ \vdots & & & & & & \\ M_{p_r^{n_r}} & = & \boxed{R/(p_r^{\nu_1^{(r)}})} & \oplus & \cdots\cdots & \oplus & \boxed{R/(p_r^{\nu_s^{(r)}})} \\ & & \cong R/(q_1) & & & & \cong R/(q_r) \end{array}$$

& $q_i = \prod_{j=1}^{s} p_j^{\nu_i^{(j)}}$ $(p^0 = 1)$

<u>Claim</u> $R/(p_1^{\nu_i^{(1)}}) \oplus \cdots \oplus R/(p_r^{\nu_i^{(r)}}) \cong R/(q_i)$

Pf/ CRT $p_1, \dots, p_r$ are distinct coprimes so, after ignoring the 0-summands in (LHS), we get pairwise coprime ideals $(p_1^{\nu_i^{(1)}}), \dots, (p_r^{\nu_i^{(r)}})$ (Lecture 29)

$$\Rightarrow (q_i) \underset{\substack{\text{PID} \\ \text{unique factoriz.}}}{=} \bigcap_{j=1}^{r} (p_j^{\nu_i^{(j)}}) \underset{\text{Lecture 26 + HW9}}{=} \prod_{j=1}^{r} (p_j^{\nu_i^{(j)}})$$

- Iso in claim follows from CRT (Lecture 19)

By construction $q_r \mid q_{r-1} \mid \cdots \mid q_1$ because $\nu_i^{(j)} \geq \nu_{i+1}^{(j)}$ $\forall j$

<u>Uniqueness</u> $\mathrm{Ann}(M) = (q_1)$ Pick $x \in M$ with $\mathrm{Ann}(x) = q_1$

$\rightsquigarrow \mathrm{Ann}(M/(x)) = (q_2)$, etc.