Lecture 35: Bilinear forms, tensor products

Recall: Last time we define tensor products & duals of vector spaces

Def The dual of V, denoted by V^*, is defined as: $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$

Theorem 1: If V is finite-dimensional, then $\dim V^* = \dim V$. ($\underset{\text{basis}}{\mathcal{B}} \rightsquigarrow \underset{\text{dual basis}}{\mathcal{B}^*}$)

Q: How to dualize a map?

Lemma: If $f: V \longrightarrow W$ is a linear map, then $f^*: W^* \longrightarrow V^*$ defined by $f^*(\xi) = \xi \circ f: V \longrightarrow W \longrightarrow \mathbb{K}$ $\forall \xi \in W^*$ is $\mathbb{K}$-linear

More precisely: $f^*(\xi)(v) = \xi(f(v))$ $\forall v \in V$, $\forall \xi \in W^*$

Prop: (HW 11) If $\dim V = n$, $\dim W = m$ then $[f^*]_{(\mathcal{B}_W)^*(\mathcal{B}_V)^*} = [f]^T_{\mathcal{B}_V \mathcal{B}_W}$.

Theorem 2 (HW 11)

double dual

$$\begin{array}{rcll} V & \xrightarrow{\varphi} & (V^*)^* & \text{linear} \quad ; \text{ surj} \iff \dim V < \infty \\ v & \longmapsto & \varphi(v): (f \longmapsto f(v)) & \forall f \in V^* \end{array}$$

§1. Bilinear Maps and Tensor Products:

Let V_1, V_2, W be 3 vector spaces over $\mathbb{K}$

Def: A bilinear map $f: V_1 \times V_2 \longrightarrow W$ is a set map which is linear in each coordinate, ie:

- $\forall v_1 \in V_1$: $\quad \begin{array}{rcl} V_2 & \longrightarrow & W \\ w & \longmapsto & f(v_1, w) \end{array} \quad \in \mathrm{Hom}_{\mathbb{K}}(V_2, W)$ linear!
- $\forall v_2 \in V_2$: $\quad \begin{array}{rcl} V & \longrightarrow & W \\ w & \longmapsto & f(w, v_2) \end{array} \quad \in \mathrm{Hom}_{\mathbb{K}}(V_1, W)$ linear!

Obs: $f(a_1v_1 + b_1v_1', a_2v_2 + b_2v_2') \underset{\text{linear in 1st coord}}{=} a_1 f(v_1, a_2v_2 + b_2v_2') + b_1 f(v_1', a_2v_2 + b_2v_2')$

$\underset{\text{linear in 2nd coord}}{=} a_1a_2 f(v_1, v_2) + a_1b_2 f(v_1, v_2') + a_2b_1 f(v_1', v_2) + a_2b_2 f(v_1', v_2')$.

<u>Conclusion</u> f is completely determine by the values on $f(v_1, v_2)$

with $v_1 \in B_{V_1}$ = basis for V_1

$v_2 \in B_{V_2}$ = ______ V_2

If V_1 & V_2 are finite dimensional & $W = \mathbb{R}$, we will be able to build a matrix encoding the bilinear form f.

<u>Ex</u>: $f: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ is defined as:

$$f(\underline{x}, y) = x^T \underbrace{\Big[f(e_i, e_j) \Big]_{i,j}}_{n \times m} y$$

<u>Def</u>: The <u>Tensor product</u> $V_1 \otimes_{\mathbb{K}} V_2$ is a vector space together with a bilinear map

$$V_1 \times V_2 \xrightarrow{\varphi} V_1 \otimes_{\mathbb{K}} V_2$$

$$(v_1, v_2) \longmapsto v_1 \otimes v_2 \leftarrow \text{indecomposable tensor}$$

satisfying the following <u>universal property</u>: For any vector sp. W over $\mathbb{K}$ and any bilinear map $f: V_1 \times V_2 \longrightarrow W$, there exists a unique $\tilde{f}: V_1 \otimes V_2 \longrightarrow W$ linear map making the following diagram commute:

$$(*) \quad \begin{tikzcd} V_1 \times V_2 \arrow[r, "f"] \arrow[d, "\varphi"'] & W \\ V_1 \otimes_{\mathbb{K}} V_2 \arrow[ur, dashed, "\exists!\, \tilde{f} \text{ linear}"'] & \end{tikzcd}$$

<u>Idea</u>: Build a space so that $\tilde{f}$ is linear.

<u>Obs</u>: The pair $(V_1 \otimes_{\mathbb{K}} V_2, \tilde{f})$ will be unique up to unique iso. (HW 11)

<u>Example</u>: $V \otimes_{\mathbb{K}} \mathbb{K} \cong V$ $\quad v \otimes 1 \longleftrightarrow v$. $\quad \varphi(v,1) = v \otimes 1 = v$

$\tilde{f}(v) = f(v,1)$

Construction: $V_1 \otimes_{\mathbb{K}} V_2$ is a vector space spanned by the set (a basis!) $B = \{(v_1, v_2) : v_1 \in V_1, v_2 \in V_2\}$ & quotiented by a subspace H

- $$V_1 \otimes_{\mathbb{K}} V_2 = \frac{\bigoplus_{v_1 \in V_1, v_2 \in V_2} \mathbb{K}(v_1, v_2)}{H}$$

H = v. sp generated by:

(1) $(z_1 v_1 + z_1' v_1', v_2) - z_1(v_1, v_2) - z_1'(v_1', v_2)$

(2) $(v_1, z_2 v_2 + z_2' v_2') - z_2(v_1, v_2) - z_2'(v_1, v_2')$

Notation: $\overline{(v_1, v_2)} = v_1 \otimes v_2$ in $V_1 \otimes V_2$

So in the quotient: $(z_1 v_1 + z_1' v_1') \otimes v_2 = z_1(v_1 \otimes v_2) + z_1'(v_1' \otimes v_2)$.

$(v_1 \otimes (z_2 v_2 + z_2' v_2') = z_2(v_1 \otimes v_2) + z_2'(v_1 \otimes v_2'$

Set $\varphi: V_1 \times V_2 \xrightarrow{\pi} V_1 \otimes V_2$ as the natural projection

$(v_1, v_2) \longrightarrow \overline{(v_1, v_2)} = v_1 \otimes v_2$

φ is bilinear by (1) & (2).

(1') $\varphi(z_1 v_1 + z_1' v_1', v_2) \stackrel{?}{=} z_1 \varphi(v_1, v_2) + z_1' \varphi(v_1', v_2)$ this is (1)!

(2') $\varphi(v_1, z_2 v_2 + z_2' v_2') \stackrel{?}{=} z_2 \varphi(v_1, v_2) + z_2' \varphi(v_1, v_2')$ —— (2)!

⚠ The map is bilinear because we defined $V_1 \otimes_{\mathbb{K}} V_2$ as a quotient of vector spaces. (quotient of $\mathbb{K}$-modules)

- Q: How to define $\tilde{f}$?

Use $\tilde{f}(v_1 \otimes v_2) = \tilde{f}(\varphi(v_1, v_2)) = f(v_1, v_2)$

- $\{v_1 \otimes v_2\}$ span, so $\tilde{f}$ will be unique by construction.

We can use the universal property of $\oplus$ to get a ! linear map $g : \bigoplus_{\substack{v_1 \in V_1 \\ v_2 \in V_2}} \mathbb{K}(v_1, v_2) \longrightarrow W$ with $g(v_1, v_2) = f(v_1, v_2)$
+ extend linearly
($\{(v_1, v_2)\}$ is a basis for this vector space)

Now we want to check $g|_H = 0$.

(1'') $g((z_1 v_1 + z_1' v_1', v_2)) = f((z_1 v_1 + z_1' v_1', v_2))$
$\underset{f \text{ bilinear}}{=} z_1 f(v_1, v_2) + z_1' f(v_1', v_2) = z_1 g(v_1, v_2) + z_1' g(v_1', v_2)$

So $g((z_1 v_1 + z_1' v_2', v_2) - z_1 (v_1, v_2) - z_1' (v_1', v_2)) = 0$

(2'') Similarly $g((v_1, z_2 v_2 + z_2' v_2) - z_2 (v_1, v_2) - z_2' (v_1, v_2')) = 0$

Conclusion: $H \leq \ker g$ so g gives a unique linear map

$$\tilde{f} : V_1 \otimes V_2 = \frac{\bigoplus_{v_i \in V_i} \mathbb{K}(v_1, v_2)}{H} \longrightarrow W.$$

- The diagram (*) commutes by construction. $\square$