Lecture 36: Tensor products, Hom tensor adjointness

Recall: The tensor product $V_1 \otimes_{\mathbb{K}} V_2$ is a vector space together with a bilinear map

$$\begin{aligned} V_1 \times V_2 &\xrightarrow{\varphi} V_1 \otimes_{\mathbb{K}} V_2 \\ (v_1, v_2) &\longmapsto v_1 \otimes v_2 \end{aligned}$$

$\leftarrow$ indecomposable tensor

satisfying the following universal property: For any vector sp. W over $\mathbb{K}$ and any bilinear map $f: V_1 \times V_2 \longrightarrow W$, there exists a unique $\tilde{f}: V_1 \otimes V_2 \longrightarrow W$ linear map making the following diagram commute:

$$\begin{array}{c} (v_1, v_2) \\ \downarrow (*) \\ v_1 \otimes v_2 = \overline{(v_1, v_2)} \end{array}
\qquad
\begin{tikzcd} V_1 \times V_2 \arrow[r, "f"] \arrow[d, "\varphi"'] & W \\ V_1 \otimes_{\mathbb{K}} V_2 \arrow[ur, dashed, "\exists ! \tilde{f} \text{ linear}"'] & \end{tikzcd}$$

Defined $V_1 \otimes_K V_2 = \dfrac{\displaystyle\bigoplus_{\substack{v_1 \in V_1 \\ v_2 \in V_2}} \mathbb{K}(v_1, v_2)}{H}$ $\leftarrow$ vector space with basis $B = \{(v_1, v_2) : v_1 \in V_1 \text{ \& } v_2 \in V_2\}$

& H = relations we expect on $V_1 \otimes_{\mathbb{K}} V_2$ = subspaces generated by ① & ② for all $v_1, v_1' \in V_1$, $v_2, v_2' \in V_2$ & $a_1, b_1, a_2, b_2 \in \mathbb{K}$:

① $(a_1 v_1 + b_1 v_1', v_2) - a_1(v_1, v_2) - b_1(v_1', v_2)$

($\leadsto$ $(a_1 v_1 + b_1 v_1') \otimes v_2 = a_1(v_1 \otimes v_2) + b_1(v_1' \otimes v_2)$ in $V_1 \otimes_{\mathbb{K}} V_2$

② $(v_1, a_2 v_2 + b_2 v_2') - a_2(v_1, v_2) - b_2(v_1, v_2')$

($\leadsto$ $v_1 \otimes (a_2 v_2 + b_2 v_2') = a_2(v_1 \otimes v_2) + b_2(v_1 \otimes v_2')$ in $V_1 \otimes_{\mathbb{K}} V_2$

- $\tilde{f}$ is defined as $\tilde{f}(v_1 \otimes v_2) = f(v_1, v_2)$
- We check it is well-defined by setting $g: \bigoplus_{\substack{v_1 \in V_1 \\ v_2 \in V_2}} \mathbb{K}(v_1, v_2) \longrightarrow W$

to be the unique $\mathbb{K}$-linear map with $\delta(\underbrace{(v_1,v_2)}_{\text{they form a basis.}}) = f(v_1,v_2)$ &

using the fact that f is bilinear to check $\delta|_H = 0$ $\quad (\delta(1) = \delta(2) = 0)$

So δ determines a $\mathbb{K}$-linear map on $V_1 \otimes V_2 = \dfrac{\bigoplus_{v_1\in V_1, v_2 \in V_2} \mathbb{K}(v_1,v_2)}{H} \longrightarrow W$

& this is precisely $\tilde{f}$. $\square$

Proposition: If $f_1 : V_1 \longrightarrow W_1$ & $f_2 : V_2 \longrightarrow W_2$ are $\mathbb{K}$-linear, then

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "(f_1,f_2)"] \arrow[dr, dashed, "\text{bilinear}"'] & W_1 \times W_2 \arrow[d, "\text{bilinear}"] \\ & W_1 \otimes W_2 \end{tikzcd}$$

$\rightsquigarrow$ get! $f_1 \otimes f_2 : V_1 \otimes V_2 \longrightarrow W_1 \otimes W_2$ linear.

§2 Bases for $V_1 \otimes V_2$:

Lemma: $\dim_{\mathbb{K}}(V_1 \otimes V_2) = \dim_{\mathbb{K}} V_1 \cdot \dim_{\mathbb{K}} V_2$ (product of cardinalities)

Pf/ We write down a basis for $V_1 \otimes V_2$ as a product of bases.

If $B_{V_1} = \{v_i^{(1)} : i \in I_1\}$ is a basis for V_1 &

$B_{V_2} = \{v_i^{(2)} : i \in I_2\}$ ———— V_2, then

Claim $B = \{ v_i^{(1)} \otimes v_j^{(2)} \quad i \in I_1, j \in I_2 \}$ is a basis for $V_1 \otimes V_2$

- B spans $V_1 \otimes V_2$:

It's enough to write the spanning set of indecomposable tensors as (finite) linear combinations of elements in B.

If $v_1 \in V_1$, $v_2 \in V_2$ then:

$$v_1 = \sum_{i\in I_1} a_i v_i^{(1)} \quad \& \quad v_2 = \sum_{j \in I_2} b_j v_j^{(2)} \quad \text{with } a_i = 0,\ b_j = 0$$

for all but finitely many $i \in I_1$, $j \in I_2$.

Then $v_1 \otimes v_2 = \varphi\Big(\sum_{\substack{i\in I_1\\ \text{finite}}} a_i v_i^{(1)}, \sum_{\substack{j\in I_2\\ \text{finite}}} b_j v_j^{(2)}\Big)$

$$\underset{\text{Reln ①}}{=} \sum_{\substack{i\in I_1\\ \text{finite}}} a_i \varphi\Big(v_i^{(1)}, \sum_{j\in I_2} b_j v_j^{(2)}\Big)$$

$$\underset{\text{Reln ②}}{=} \sum_{\substack{i\in I_1\\ j\in I_2\\ \text{finite}}} \underbrace{a_i b_j}_{\in \mathbb{K}} \underbrace{\varphi(v_i^{(1)}, v_j^{(2)})}_{= v_i^{(1)} \otimes v_j^{(2)}}$$

$a_i b_j = 0$ for all but finitely many $i\in I_1$, $j \in I_2$.

- <u>B is l.i.:</u>

$$\sum_{\substack{i\in I_1\\ j\in I_2\\ \text{finite}}} c_{ij}\, v_i^{(1)} \otimes v_j^{(2)} = 0 \in V_1 \otimes V_2$$

<u>want to show</u> $c_{ij} = 0$

Use def to rewrite it as

$$\sum_{\substack{j\in I_2\\ \text{finite}}} \Big(\underbrace{\sum_{\substack{i\in I_1\\ \text{finite}}} c_{ij} v_i^{(1)}}_{\in V_1}\Big) \otimes \underbrace{v_j^{(2)}}_{\in V_2} = 0. \quad (**)$$

Pick $l \in I_2$. <u>We'll show</u> $c_{il} = 0 \ \forall i$

Since B_{V_2} is a basis for V_2: $\exists\ \psi : V_2 \to \mathbb{K}$ linear with

$$\psi(v^{(2)}_j) = \delta_{j,l} \qquad \psi = (v_l^{(2)})^* \in V_2^*.$$

By Proposition $\exists\ id_{V_1} \otimes \psi : V_1 \otimes V_2 \to \underbrace{V_1 \otimes_{\mathbb{K}} \mathbb{K}}_{= V_1}$ linear with

$$(id_{V_1} \otimes \psi)(v_1 \otimes v_2) = v_1 \otimes \psi(v_2).$$

Apply $id_{V_1} \otimes \psi$ to $(**)$. Then:

$$0 = (id_{V_1} \otimes \psi)(0) = \sum_{\substack{j\in I_2\\ \text{finite}}} \Big(\underbrace{\sum_{\substack{i\in I_1\\ \text{finite}}} c_{ij} v_i^{(1)}}_{\in V_1}\Big) \otimes \underbrace{\psi(v_j^{(2)})}_{\in \mathbb{K}}$$

$0 \in V_1 \otimes_{\mathbb{K}} \mathbb{K} = V_1$

$\vec{v} \otimes a \leftrightarrow a\vec{v}$

$$= \sum_{i \in I_1} c_{il}\, v_i^{(1)} \otimes 1$$

only $j = l$ survives

$$\Rightarrow \underset{\substack{\uparrow \\ V_1}}{0} = \sum_{\substack{i \in I_1 \\ \text{finite}}} c_{il}\, v_i^{(1)} \underset{B_{V_1} \text{ basis}}{\Longrightarrow} c_{il} = 0 \quad \forall i$$

$\square$

Remark: Assume $\dim V_i = n_i < \infty$ & $\dim W_i = m_i < \infty$.

Assume $f_1 : V_1 \longrightarrow W_1$ linear are identified with matrices

$f_2 : V_2 \longrightarrow W_2$ —

$X_1 \in \mathrm{Mat}_{m_1 \times n_1}(\mathbb{K})$ & $X_2 \in \mathrm{Mat}_{m_2 \times n_2}(\mathbb{K})$. Then: $f_1 \otimes f_2$ gets identified with a matrix $X_1 \otimes X_2 \in \mathrm{Mat}_{(m_1 \cdot m_2) \times (n_1 \cdot n_2)}(\mathbb{K})$.

More precisely, if $X_1 = \begin{bmatrix} a_{11} & \cdots & a_{1n_1} \\ \vdots & & \\ a_{m_1 1} & \cdots & a_{m_1 n_1} \end{bmatrix}$ & $X_2 = \begin{bmatrix} b_{11} & \cdots & b_{1n_2} \\ \vdots & & \\ b_{m_2 1} & \cdots & b_{m_2 n_2} \end{bmatrix}$,

then $X_1 \otimes X_2 = \begin{bmatrix} a_{11} X_2 & \cdots & a_{1n_1} X_2 \\ \vdots & & \\ \boxed{a_{m_1 1} X_2} & \cdots & a_{m_1 n_1} X_2 \end{bmatrix}$

$m_2 \times n_2$ matrix

$B_{V_1} = \{ v_i^{(1)} \ i = 1, \dots, n_1 \}$

$B_{V_2} = \{ v_i^{(2)} \ i = 1, \dots, n_2 \}$

$B_{W_1} = \{ w_i^{(1)} \ i = 1, \dots, m_1 \}$

$B_{W_2} = \{ w_i^{(2)} \ i = 1, \dots, m_2 \}$

Here we use $B_{V_1 \otimes V_2} = \text{"} B_{V_1} \times B_{V_2} \text{"} = \bigcup_{i=1}^{n_1} \{ v_i^{(1)} \otimes v_j^{(2)} : j = 1, \dots, n_2 \}$

$B_{W_1 \otimes W_2} = \text{"} B_{W_1} \times B_{W_2} \text{"} = \bigcup_{i=1}^{m_1} \{ w_i^{(1)} \otimes w_j^{(2)} : j = 1, \dots, m_2 \}$

(as in the Lemma)

§3. Hom-Tensor adjointness:

Prop: There is a natural map $V^* \otimes W \xrightarrow{\Phi} \mathrm{Hom}(V,W)$

with $\Phi(\xi \otimes w) : v \longmapsto \underbrace{\xi(v)}_{\in \mathbb{K}} \cdot w$.

Moreover, Φ is an isomorphism if V is finite-dimensional

In general: Φ is always injective.

Proof Define $\varphi : V^* \times W \longrightarrow \mathrm{Hom}(V,W)$

$(\xi, w) \longmapsto \{ v \mapsto \xi(v) w \}$

- Easy check: φ is bilinear. Hence, it yields a unique linear map

$$\phi : V^* \otimes W \longrightarrow \mathrm{Hom}(V,W)$$

with $\phi(\xi \otimes w) = \varphi(\xi, w)$.

- Φ is injective: Let $\alpha \in V^* \otimes W$ be such that $\phi(\alpha) = 0$

Write $\alpha = \sum_{j=1}^{N} \xi_j \otimes w_j$ (absorb scalars into ξ_j)

We can assume $w_1, \dots, w_N$ are linearly independent. Otherwise, use the dependency relation to reduce the number of w's.

(E.g. write $w_N = \sum_{j=1}^{N-1} a_j w_j \rightsquigarrow \sum_{j=1}^{N-1} \xi_j \otimes w_j + \sum_{j=1}^{N-1} \xi_N \otimes a_j w_j$

$= \sum_{j=1}^{N-1} \xi_j \otimes w_j + \sum_{j=1}^{N-1} (a_j \xi_N \otimes w_j)$

$= \sum_{j=1}^{N} \underbrace{(\xi_j + a_j \xi_N)}_{\text{new } \xi_j'} \otimes w_j$)

Then: $\forall v \in V$ $\sum_{j=1}^{N} \xi_j(v)\, w_j = \phi(\alpha)(v) = 0$

$\underset{\{w_1,\dots w_N\} \text{ li}}{\Longrightarrow}$ $\xi_j(v) = 0$ $\forall j = 1 \cdots N$

But $\xi_j(v) = 0$ $\forall v \in V$ $\Longrightarrow \xi_j = 0$

Conclude: $\alpha = \sum_{j=1}^{N} 0 \otimes w_j = 0.$

• Claim: If $\dim V = n < \infty$, then ϕ is surjective

Pf/ Let $B_V = \{v_1, \dots, v_n\}$ be a basis for V. Given $f \in \mathrm{Hom}_{\mathbb{K}}(V, W)$

let $w_i = f(v_i)$ $i = 1, \dots n$.

Then $\alpha = \sum_{i=1}^{m} v_i^* \otimes w_i \in V^* \otimes W$ satisfies $\phi(\alpha) = f$.

because $\sum_{i=1}^{m} \phi(v_i^* \otimes w_i)_{(v_j)} = \sum_{i=1}^{m} \underbrace{v_i^*(v_j)}_{\delta_{ij}} \cdot w_i = w_j$

So $\phi(\alpha)$ & f agree on B_V, so they are the same function. $\square$

In particular, when $W = V$ & $\dim V < \infty$, we use this to write a canonical tensor in $V^* \otimes V$, that is basis independent (HW12)