Lecture 37: Tensor, Exterior & Symmetric Algebras

Given \( V_1, \ldots, V_n \) \& \( W \) \( K \)-vector spaces, we can define multilinear maps 
\[ f : V_1 \times \cdots \times V_n \longrightarrow W \] as linear when restricted to each word.

We construct a tensor product \( V_1 \otimes_K V_2 \otimes_K \cdots \otimes_K V_n \):

\begin{itemize}
  \item \textbf{inductively:} \( V_1 \otimes_K (V_2 \otimes \cdots \otimes_K V_n) \) \quad (n \geq 2)
  \item \textbf{via universal property:} \( V_1 \otimes_K V_2 \otimes_K \cdots \otimes_K V_n \) is a \( K \)-vector space together with a multilinear map \( \Psi : V_1 \times \cdots \times V_n \longrightarrow \bigotimes_{i=1}^n V_i \) 
  \[ (v_1, \ldots, v_n) \longmapsto v_1 \otimes \cdots \otimes v_n \]
\end{itemize}

satisfying: whenever we are given a multilinear map \( f : V_1 \times \cdots \times V_n \longrightarrow W \) we can find a unique linear map \( \overline{f} : \bigotimes_{i=1}^n V_i \longrightarrow W \) making the diagram commute.

\[
\begin{array}{ccc}
V_1 \times \cdots \times V_n & & \longrightarrow & W \\
\downarrow \Psi & & & \downarrow \overline{f} \\
V_1 \otimes_K \cdots \otimes_K V_n & & \longrightarrow & \bigotimes_{i=1}^n V_i \\
\end{array}
\]

Basis for \( V_1 \otimes \cdots \otimes V_n = \{ v_1^{(1)} \otimes \cdots \otimes v_n^{(n)} : v_1^{(1)} \in B_1, \ldots, v_n^{(n)} \in B_n \} \)
& \( B_i \) is a basis for \( V_i \) for \( i = 1, \ldots, n \).

\( \S 1. \) Tensor Algebra:

Pick any \( K \)-vector space \( V \). Inductively we define
\[ T^k(V) = V \otimes_k \cdots \otimes_k V \text{ for all } k \geq 2 \]

Collect all these tensors into a vector space via \( \oplus \)
\[ T(V) = \bigoplus_{n \geq 0} T^n(V) \quad (T^0(V) = K) \]
Prop: $T^\cdot(V)$ is a $K$-algebra $\Rightarrow$ $K$-vector space + ring structure

Proof: Enough to define the multiplication between $T^\cdot(V) \times T^\cdot(V)$

$$T^\cdot(V) \times T^\cdot(V) \longrightarrow T^{\cdot+\cdot}(V) \text{ bilinear}$$

We define it on indecomposable tensors & extend $K$-bilinearly.

$$(v, \otimes \cdots \otimes v_n) \cdot (w, \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m$$

Well-defined? (1) We can work with bases & extend $K$-linearly

2. Universal Property: $(V \times \cdots \times V) \times (V \times \cdots \times V) \xrightarrow{\text{mult}} T^\cdot(V)$

$$\Rightarrow T^\cdot(V) \times T^\cdot(V) \longrightarrow T^\cdot(V)$$

$$\text{mult} = \Psi \text{ bilinear in def of} \Psi$$

$$T^\cdot(V) \times T^\cdot(V) \xrightarrow{\text{mult}} T^{\cdot+\cdot}(V)$$

Example: $V = K^n$, then $T^\cdot(V) \leftrightarrow$ homogeneous degree $n$ polynomials in $n$ non-commuting variables.

Variables $x_i \leftrightarrow e_i$ (standard basis for $K^n$)

$$(\sum a_i e_i) \otimes (\sum b_j e_j) = \sum a_i b_j (e_i \otimes e_j)$$

$\Rightarrow T^2(K^n) = \text{ degree } polyn$ in $n$ non-commuting variables.

§ 2. Symmetric & Exterior Algebras

Next, we define two quotients of $T^k(V)$ that allow us to swap entries in tensors, with/without a sign. We assume $\det K \neq 2$
Lemma: For all \( q \in S_k \), \( S_k(V) = \bigoplus_{\lambda \in S_k} \mathbb{K} \), where \( \mathbb{K} \) is a \( k \)-dimensional vector space.

Definition: \( S_k(V) \) is the subspace spanned by \( \mathbb{K} \).

Theorem: \( S_k(V) = \bigoplus_{\lambda \in S_k} \mathbb{K} \), where \( \mathbb{K} \) is a \( k \)-dimensional vector space.

Proof: By induction on \( k \). For \( k = 1 \), it is the definition of \( N(V) \).

Base case \( k = 1 \): Use the definition of \( N(V) \).

Inductive step: Assume \( S_k(V) = \bigoplus_{\lambda \in S_k} \mathbb{K} \) for some \( k \geq 1 \). For \( k + 1 \), consider \( S_{k+1}(V) \) and use the inductive hypothesis.

Lemma: For all \( q \in S_k \), \( S_k(V) = \bigoplus_{\lambda \in S_k} \mathbb{K} \), where \( \mathbb{K} \) is a \( k \)-dimensional vector space.

Definition: \( S_k(V) \) is the subspace spanned by \( \mathbb{K} \).

Theorem: \( S_k(V) = \bigoplus_{\lambda \in S_k} \mathbb{K} \), where \( \mathbb{K} \) is a \( k \)-dimensional vector space.

Proof: By induction on \( k \). For \( k = 1 \), it is the definition of \( N(V) \).

Base case \( k = 1 \): Use the definition of \( N(V) \).

Inductive step: Assume \( S_k(V) = \bigoplus_{\lambda \in S_k} \mathbb{K} \) for some \( k \geq 1 \). For \( k + 1 \), consider \( S_{k+1}(V) \) and use the inductive hypothesis.
A typical summand of an element in $\wedge^k(V)$ is written as $v_1 \wedge \ldots \wedge v_k$ (order matters!)

- $v_1 \wedge \ldots \wedge v_k = 0$ if $v_i = v_j$ for some $i \neq j$ (by Lemma)
  - $\text{sign}(i,j) = -1$, so flipping signs introduces a negative sign.
- $v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(k)} = \text{sign}(\sigma) v_1 \wedge \ldots \wedge v_k \quad \forall \sigma \in S_k$. 

Prop: Basis for $S^k(V)$ & $\wedge^k(V)$:

If $\{v_i : i \in I\}$ is a basis for $V$ & $I$ is totally ordered, then

1. $\{v_{i_1} \wedge \ldots \wedge v_{i_s} : i_1 + \ldots + i_s = k, s \geq 1, i_j \geq 1\}$ is a basis for $S^k(V)$

2. $\{v_{i_1} \wedge \ldots \wedge v_{i_k} : 1 < i_2 < \ldots < i_k \in I\}$ is a basis for $\wedge^k(V)$

Obs: Total order if $I$ is infinite requires the axiom of choice.

Corollary: Assume dim $V = n$. Then

- dim $S^k(V) = \binom{n+k-1}{k}$ (# monomials of degree $k$ in $n$ variables)

- dim $(\wedge^k(V)) = \binom{n}{k}$ ($=0$ if $k > n$)

We can define symmetric and exterior algebras:

$\text{Sym}^* (V) = S^* (V) = \bigoplus_{n \geq 0} S^n (V)$ $\quad (S^0(V) = K, S^1(V) = V)$

$\wedge^* (V) = \bigoplus_{n \geq 0} \wedge^n (V)$ $\quad (\wedge^0(V) = K, \wedge^1(V) = V)$