Lecture 37: Tensor, Exterior & Symmetric Algebras

Given $V_1, \ldots, V_n$ & W $\mathbb{K}$-vector spaces, we can define multilinear maps $f: V_1 \times \cdots \times V_n \longrightarrow W$ as linear when restricted to each coord.

$\rightsquigarrow$ We construct a tensor product $V_1 \otimes_{\mathbb{K}} V_2 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V_n$:

- inductively: $V_1 \otimes_{\mathbb{K}} (V_2 \otimes \cdots \otimes_{\mathbb{K}} V_n)$ $\quad (n \geq 2)$

or

- Via universal property: $V_1 \otimes_{\mathbb{K}} V_2 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V_n$ is a $\mathbb{K}$-vector space

together with a multilinear map $\varphi: V_1 \times \cdots \times V_n \longrightarrow \bigotimes_{i=1}^{n} V_i$

$$(v_1, \ldots, v_n) \longmapsto v_1 \otimes \cdots \otimes v_n$$

satisfying: whenever we are given a multilinear map $f: V_1 \times \cdots \times V_n \longrightarrow W$ we can find a unique linear map $\bar{f}: \bigotimes_{i=1}^{n} V_i \longrightarrow W$ making the diagram

$$\begin{tikzcd} V_1 \times \cdots \times V_n \arrow[r, "f"] \arrow[d, "\varphi"'] & W \\ V_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V_n \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

commute.

. Basis for $V_1 \otimes \cdots \otimes V_n = \{ v^{(1)}_{i_1} \otimes \cdots \otimes v^{(n)}_{i_n} : v^{(1)}_{i_1} \in B_1, \ldots, v^{(n)}_{i_n} \in B_n \}$ & B_i is a basis for V_i for $i = 1, \ldots, n$.

§1. Tensor Algebra:

Pick any $\mathbb{K}$-vector space V. Inductively we define

$$T^k(V) = V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ terms}} \quad \text{for all } k \geq 2.$$

$\rightsquigarrow$ Collect all these tensors into 1 vector space via $\oplus$.

$$T^\bullet(V) = \bigoplus_{n \geq 0} T^n(V) \qquad (T^0(V) = \mathbb{K})$$

Prop: $T^{\bullet}(V)$ is a $\mathbb{K}$-algebra $=$ $\mathbb{K}$-vector space + ring structure

Proof: Enough to define the multiplication between $T^n(V)$ & $T^m(V)$

$$T^n(V) \times T^m(V) \longrightarrow T^{n+m}(V) \quad \underline{\text{bilinear}}$$

- We define it on indecomposable tensors & extend $\mathbb{K}$-bilinearly.

$$(v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m$$

- Well-defined? (1) We can work with bases & extend $\mathbb{K}$-linearly

(2) Universal Property:

$$\begin{tikzcd}
(\overbrace{V \times \cdots \times V}^{n \text{ copies}}) \times (\overbrace{V \times \cdots \times V}^{m \text{ copies}}) \arrow[r, "\text{mult}"] \arrow[d, "(\varphi_n, \varphi_m)"] & T^{n+m}(V) \\
T^n(V) \times T^m(V) \arrow[ur, dashed, "\varphi"] \arrow[d, "\varphi"] & \\
T^n(V) \otimes T^m(V) \arrow[uur, "="'] &
\end{tikzcd}$$

$\Rightarrow T^n(V) \times T^m(V) \longrightarrow T^{n+m}(V)$

mult $= \varphi$ bilinear in def of

$T^n(V) \otimes T^m(V) = T^{n+m}(V)$

Example: $V = \mathbb{K}^n$, then $T^m(V) \longleftrightarrow$ homogeneous degree m polynomials in n non-commuting variables.

- variables $\quad x_i \longleftrightarrow e_i \quad$ (standard basis for $\mathbb{K}^n$)

Eg. $\left(\sum_i a_i e_i\right) \otimes \left(\sum_j b_j e_j\right) = \sum_{i,j} a_i b_j \, (e_i \otimes e_j)$

$\updownarrow$

$x_i \cdot x_j$ monomial

$\rightsquigarrow T^2(\mathbb{K}^n) =$ homog degree polys in n non-commuting variables.

§2. Symmetric & Exterior Algebras

Next, we define two quotients of $T^k(V)$ that allow us to swap entries on tensors, with/without a sign. We assume char $\mathbb{K} \neq 2$

Definition:

- $S^k(V) = \mathrm{Sym}^k(V) = V^{\otimes k} \Big/ \left\langle \begin{array}{l} \text{subspace spanned by} \\ v_1 \otimes \cdots \otimes v_k - v_1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ i}}{v_{i+1}} \otimes \underset{\substack{\uparrow \\ i+1}}{v_i} \otimes \cdots \otimes v_k \\ \quad 1 \leq i \leq n-1 \\ \text{for } v_1, \ldots, v_k \in V \end{array} \right\rangle$

  (k^{th} symmetric product of V)

- $\Lambda^k(V) = V^{\otimes k} \Big/ \left\langle \begin{array}{l} \text{subspace spanned by} \\ v_1 \otimes \cdots \otimes v_k + v_1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ i}}{v_{i+1}} \otimes \underset{\substack{\uparrow \\ i+1}}{v_i} \otimes \cdots \otimes v_k \\ \quad 1 \leq i \leq n-1 \\ \text{for } v_1, \ldots, v_k \in V \end{array} \right\rangle$

  (k^{th} exterior (or alternating) product of V)

Obs 1: A typical summand of an element in $S^k(V)$ is written as $v_1 \cdots v_k$

- $S^k(\mathbb{K}^n)$ = deg k homogeneous polynomials in n commuting variables!

Eg: $S^2(\mathbb{K}^n) \ni \left(\sum_{i=1}^n a_i e_i\right) \otimes \left(\sum_{j=1}^n b_j e_j\right) = \sum_{i,j=1}^n a_i b_j \, e_i \otimes e_j$

$$= \sum_{i<j} (a_i b_j + a_j b_i)\, e_i e_j + \sum_{i=1}^n a_i b_i \, e_i e_i$$

$e_i \otimes e_j = e_j \otimes e_i$ in $S^2(\mathbb{K}^n)$

$$\longleftrightarrow \sum_{i<j} (a_i b_j + a_j b_i)\, x_i x_j + \sum_{i=1}^n a_i b_i \, x_i^2 .$$

Lemma: $v_1 \otimes \cdots \otimes v_k - \mathrm{sg}(\sigma)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} = 0$ in $\Lambda^k V$ for all $\sigma \in S_k$, $v_1, \ldots, v_n \in V$

Pf/ By induction on $\mathrm{len}(\sigma)$ (write σ as a product of simple transpositions)

Base case $\mathrm{len}(\sigma) = 1$: is the definition of $\Lambda^k(V)$. $\square$

Obs 2: A typical summand of an element in $\Lambda^k(V)$ is written as $v_1 \wedge \cdots \wedge v_k$ (order matters!)

- $v_1 \wedge \cdots \wedge v_k = 0$ if $v_i = v_j$ for some $i \neq j$ (by Lemma)

sign $(ij) = -1$, so flipping signs introduces a negative sign

- $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \text{sign}(\sigma)\,(v_1 \wedge \cdots \wedge v_k) \quad \forall \sigma \in S_k$.

Prop: Basis for $S^k(V)$ & $\Lambda^k(V)$:

If $\{v_i : i \in I\}$ is a basis for V & I is totally ordered, then

(1) $\{v_{i_1}^{r_{i_1}} \cdots v_{i_s}^{r_{i_s}} : r_{i_1} + \cdots + r_{i_s} = k,\ s \geq 1,\ r_{i_j} \geq 1\}$ is a basis for $S^k(V)$

(2) $\{v_{i_1} \wedge \cdots \wedge v_{i_k} \quad i_1 < i_2 < \cdots < i_k \text{ in } I\}$ is a basis for $\Lambda^k(V)$

Obs: Total order if I is infinite requires the axiom of choice.

Corollary: Assume $\dim V = n$. Then,

(1) $\dim S^k V = \binom{n+k-1}{k}$ (# monomials of deg k in n variables)

(2) $\dim(\Lambda^k V) = \binom{n}{k}$ ($=0$ if $k > n$)

We can define Symmetric and exterior algebras:

$$\text{Sym}^\bullet(V) = S^\bullet(V) = \bigoplus_{n \geq 0} S^n(V) \qquad (S^0(V) = \mathbb{K},\ S^1(V) = V)$$

$$\Lambda^\bullet(V) = \bigoplus_{n \geq 0} \Lambda^n(V) \qquad (\Lambda^0(V) = \mathbb{K},\ \Lambda^1(V) = V)$$