Lecture 38: Exterior & Symmetric Algebras

Recall: Given a vector space V over a field $\mathbb{K}$.

$$T^n(V) = V^{\otimes n}$$

$$S^n(V) = V^{\otimes n} \Big/ \left\langle v_1 \otimes \cdots \otimes v_n - v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n \; : \; \begin{smallmatrix} 1 \leq i \leq n-1 \\ v_j \in V \end{smallmatrix} \right\rangle$$

$$\Lambda^n(V) = V^{\otimes n} \Big/ \left\langle v_1 \otimes \cdots \otimes v_n + v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n \; : \; \begin{smallmatrix} 1 \leq i \leq n-1 \\ v_j \in V \end{smallmatrix} \right\rangle$$

Notation for $S^n(V)$: $\quad v_1 \cdots v_n = \overline{v_1 \otimes \cdots \otimes v_n}$

——— $\Lambda^n(V)$: $\quad v_1 \wedge \cdots \wedge v_n = \overline{v_1 \otimes \cdots \otimes v_n}$

§1. Tensor, exterior & symmetric algebras:

We can define Tensor, Symmetric and exterior algebras:

- $T^\bullet(V) = \bigoplus_{n \geq 0} T^n(V) \qquad (T^0(V) = \mathbb{K},\ T^1(V) = V)$
- $\mathrm{Sym}^\bullet(V) = S^\bullet(V) = \bigoplus_{n \geq 0} S^n(V) \qquad (S^0(V) = \mathbb{K},\ S^1(V) = V)$
- $\Lambda^\bullet(V) = \bigoplus_{n \geq 0} \Lambda^n(V) \qquad (\Lambda^0(V) = \mathbb{K},\ \Lambda^1(V) = V)$

Multiplication on $T^\bullet(V)$: $\quad T^n(V) \times T^m(V) \longrightarrow T^{m+n}(V)$

by extending $\mathbb{K}$-bilinearly $(v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m$

Prop: $S^\bullet(V)$ & $\Lambda^\bullet(V)$ are $\mathbb{K}$-algebras.

Pf/ Follow the same idea as in $T^\bullet(V)$. Define multiplication

$$\Phi: S^n(V) \times S^m(V) \longrightarrow S^{n+m}(V)$$

$$v_1 \cdots v_n \times w_1 \cdots w_m \longmapsto v_1 \cdots v_n w_1 \cdots w_m$$

$$\Psi: \Lambda^n(V) \times \Lambda^m(V) \longrightarrow \Lambda^{n+m}(V)$$

$$v_1 \wedge \cdots \wedge v_n \times w_1 \wedge \cdots \wedge w_m \longmapsto v_1 \wedge \cdots \wedge v_n \wedge w_1 \wedge \cdots \wedge w_m$$

More precisely: $\Phi(\overline{u}, \overline{u'}) = \overline{\varphi(u, u')}$ in $S^{n+m}(V)$

$\Psi(\overline{u}, \overline{u'}) = \overline{\varphi(u, u')}$ in $\Lambda^{n+m}(V)$

To show it's well-defined, need to show the relations defining $S^k(V)$ & $\Lambda^k(V)$ are preserved (Exercise)

. multiplication is associative, distributive by construction $\square$

§2 More on $T^\bullet(V)$, $S^\bullet(V)$, $\Lambda^\bullet(V)$:

Above description: $S^n(V)$ & $\Lambda^n(V)$ are quotients of $T^n(V)$.

Alternative approach: in char$(\mathbb{K}) = 0$ we can view $S^n(V)$ & $\Lambda^n(V)$ as subspaces of $T^n(V)$. & the multiplication respects the structure.

- Define an action of S_n on $T^n(V)$ via

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

(Need to show it extends from index tensors to $T^n(V)$, can do this via univ property $\sigma_\bullet : \underbrace{V \times \cdots \times V}_{n \text{ times}} \longrightarrow T^n(V)$ multilinear

$\rightsquigarrow \exists!\ \overline{\sigma}_\bullet : T^n(V) \longrightarrow T^n(V)$ with $\sigma(v_1, \ldots, v_n) = \overline{\sigma}(v_1 \otimes \cdots \otimes v_n)$)

- Define 2 operators $S, A : T^n V \longrightarrow T^n V$

$$S(\xi) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma(\xi)$$

$$A(\xi) = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, \sigma(\xi)$$

Prop: (1) $S^2 = S$, $A^2 = A$

(2) $\mathrm{Ker}(S) = \langle v_1 \otimes \cdots \otimes v_n - v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n \rangle$, $1 \leq i \leq n-1$, $v_1, \ldots, v_n \in V$

$\mathrm{Ker}(A) = \langle v_1 \otimes \cdots \otimes v_n + v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n \rangle$, $1 \leq i \leq n-1$, $v_1, \ldots, v_n \in V$.

So $\mathrm{Im}(S) \cong \dfrac{T^n(V)}{\mathrm{Ker}(S)} = \mathrm{Sym}^n(V)$

$\mathrm{Im}(A) \cong \dfrac{T^n(V)}{\mathrm{Ker}(A)} = \Lambda^n(V)$

Proof: See HW12.

Obs: View $\mathrm{Sym}^n(V) = \left(T^n(V)\right)^{S_n}$, $\Lambda^n(V) = \left(T^n(V)\right)^{S_n, \varepsilon}$ ← signed action!

Q. Compatibility with linear maps?

$f: V \xrightarrow[\text{linear}]{} W \rightsquigarrow$ $T^n(f): T^n(V) \longrightarrow T^n(W)$ linear

$S^n(f): S^n(V) \longrightarrow S^n(W)$ linear

minors of matrices (next time) → $\Lambda^n(f): \Lambda^n(V) \longrightarrow \Lambda^n(W)$ linear

- $T^n(f)(v_1 \otimes \cdots \otimes v_n) = f(v_1) \otimes \cdots \otimes f(v_n)$

(Via univ. prop.:

$$\begin{tikzcd}
V \times \cdots \times V \arrow[r, "f \times \cdots \times f", "\text{mult}"'] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & W \times \cdots \times W \arrow[r, "\varphi", "\text{mult}"'] & T^n(W) \\
T^n(V) \arrow[urr, dashed, "\exists!\, T^n(f)\ \text{linear}"'] & &
\end{tikzcd}$$
)

- $S^n(f)(v_1 \cdots v_n) = f(v_1) \cdots f(v_n)$
- $\Lambda^n(f)(v_1 \wedge \cdots \wedge v_n) = f(v_1) \wedge \cdots \wedge f(v_n)$

Check $T^n(f)\big|_{\substack{\ker S \\ \text{for } T^n(V)}} \subset \underset{\text{for } T^n(W)}{\mathrm{Ker}(S)}$, $\Lambda^n(f)\big|_{\substack{\mathrm{Ker}(A) \\ \text{for } T^n(V)}} \subset \underset{\text{for } T^n(W)}{\mathrm{Ker}(A)}$

Q: Universal Properties?

Prop Given a $\mathbb{K}$-algebra A (unital, associative) & a $\mathbb{K}$-linear map $V \xrightarrow{\varphi} A$, then

- $\exists$ a unique extension : $\overline{\varphi}: T^\bullet(V) \longrightarrow A$. $(\overline{\varphi}|_V = \varphi)$
- If A is a commutative algebra, then $\exists! \ \overline{\varphi}: S^\bullet(V) \longrightarrow A$
- If A is skew-commutative, ie $ab = -ba \ \forall a,b \in A$, then
  $\exists! \ \overline{\varphi} \ \Lambda^\bullet(V) \longrightarrow A$.

Prop: $\Phi \ V \otimes V \simeq S^2(V) \oplus \Lambda^2(V)$ $(\simeq$ even + odd func$)$

$$(v \otimes v') \longmapsto \left(\frac{v \otimes v' + v' \otimes v}{2} ; \frac{v \otimes v' - v' \otimes v}{2} \right)$$

Pf/ Well-defined via universal property:

Write $V \times V \longrightarrow S^2(V) \oplus \Lambda^2(V)$ bilinear

$(v, v') \longmapsto (v \cdot v' , v \wedge v')$

$\Rightarrow$ This map factors through $V \otimes V$. This defines Φ.

We view $S^2(V)$ & $\Lambda^2(V)$ as subspaces of $\Lambda \otimes \Lambda$. & construct the inverse map Φ^{-1} via the inclusions $S^2(V) \hookrightarrow T^2(V)$

$\Lambda^2(V) \hookrightarrow T^2(V)$ □

⚠ This decomposition does not extend beyond $n=2$. Instead

$$V^{\otimes n} \simeq \bigoplus_{\lambda \vdash n} \mathbb{S}^\lambda(V)$$

Schur functors

(partitions of n)
$\lambda_1 \geq \ldots \geq \ldots \geq 0$, $\sum_i \lambda_i = n$ $\lambda_i \in \mathbb{Z}_{\geq 0}$

Q: What happens to T^n, S^n & Λ^n when we consider direct sums?

Lemma: Consider 2 vector spaces V & W. Then $\forall n$:

(1) $S^n(V \oplus W) = \bigoplus_{i=0}^{n} S^i(V) \otimes S^{n-i}(W)$

(Think of polynomials in commuting variables x_i (for basis elements in V)
y_j ———————————)

(2) $\Lambda^n(V \oplus W) = \bigoplus_{i=0}^{n} \Lambda^i(V) \otimes \Lambda^{n-i}(W)$

(3) $T^n(V \oplus W) = \bigoplus_{k=0}^{n} \left(\bigoplus_{\substack{i_1+\cdots+i_k=n \\ i_1 \geq 0;\ i_2,\ldots,i_k > 0}} \left(T^{i_1}(V) \otimes T^{i_2}(W) \otimes T^{i_3}(V) \otimes \cdots \right) \right)$

(Variables don't commute, so we can't rearrange putting all of the V-part before the W-piece)

Pf/ Pick bases for V & W & check both sides of each identity share the same natural bases. (see HW 12)