Lecture 39: Determinants, minors & permanents

Recall: Given 2 vector spaces V, W over a field $\mathbb{K}$ & a linear map $V \xrightarrow{f} W$, we have 3 natural linear maps for all $n \geq 0$

(1) $T^n(V) \xrightarrow{T^n(f) = f^{\otimes n}} T^n(W)$ with $f^{\otimes n}(v_1 \otimes \cdots \otimes v_n) = f(v_1) \otimes \cdots \otimes f(v_n)$

(2) $S^n(V) \xrightarrow{S^n(f)} S^n(W)$ with $S^n(f)(v_1 \cdots v_n) = f(v_1) \cdots f(v_n)$

(3) $\Lambda^n(V) \xrightarrow{\Lambda^n(f)} \Lambda^n(W)$ with $\Lambda^n(f)(v_1 \wedge \cdots \wedge v_n) = f(v_1) \wedge \cdots \wedge f(v_n)$

(1) is defined via universal property; (2) & (3) are obtained from $T^n(f)$ by observing $f^{\otimes}\Big|_{\langle v_1 \otimes \cdots \otimes v_n \mp v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n \,:\, i=1,\ldots,n-1,\ v_1,\ldots,v_n \in V\rangle}$ lies in

$\langle w_1 \otimes \cdots \otimes w_n \mp w_1 \otimes \cdots \otimes w_{i+1} \otimes w_i \otimes \cdots \otimes w_n \,:\, i=1,\ldots,n-1,\ w_1,\ldots,w_n \in W\rangle$.

§1 Determinants & minors:

Fix V, W finite dimensional $\mathbb{K}$-vector spaces & $f: V \to W$ linear

Say $V \simeq \mathbb{K}^n$, $W \simeq \mathbb{K}^m$ (pick bases for V & W), so

$f \in \mathrm{Mat}_{m \times n}(\mathbb{K})$

$$\rightsquigarrow \quad \begin{array}{ccc} \Lambda^k(V) & \xrightarrow{\Lambda^k(f)} & \Lambda^k(W) \\ \wr\wr & \circlearrowleft & \wr\wr \\ \mathbb{K}^{\binom{n}{k}} & \xrightarrow{\quad g \quad} & \mathbb{K}^{\binom{m}{k}} \end{array}$$

$g \in \mathrm{Mat}_{\binom{m}{k} \times \binom{n}{k}}(\mathbb{K})$

basis $\{v_{j_1} \wedge \cdots \wedge v_{j_k}\}$, $\{w_{i_1} \wedge \cdots \wedge w_{i_k}\}$, $1 \leq j_1 < \cdots < j_k \leq n$, $1 \leq i_1 < \cdots < i_k \leq m$

Def: Minors of f are the entries of the matrix for g.

More precisely, write $\underline{i} = (i_1, \ldots, i_k) \in \binom{[m]}{k}$ (k-subsets of $[m]$)

$\underline{j} = (j_1, \ldots, j_k) \in \binom{[n]}{k}$ (——— $[n]$)

$[N] = \{1, \ldots, N\}$

$$\Lambda^k(f)_{\underline{i},\underline{j}} = \det\left(f^{\underline{i}}_{\underline{j}}\right) = \boxed{\Delta^{\underline{i}}_{\underline{j}}(f)}$$

↳ submatrix of f with rows in $\underline{i}$ & columns in $\underline{j}$.

Q How do we compute $\Lambda^k(f)_{\underline{i},\underline{j}}$?

Fix $B = \{v_1, \dots, v_n\}$ a basis for V
$B' = \{w_1, \dots, w_m\}$ ——— W

Write $f(v_j) = \sum_{i=1}^{n} a_{ij} w_i$

$$\Lambda^k(f)(v_{j_1} \wedge \dots \wedge v_{j_k}) = \left(\sum_{i=1}^{m} a_{ij_1} w_i\right) \wedge \dots \wedge \left(\sum_{i=1}^{m} a_{ij_k} w_i\right)$$

$\Rightarrow \Lambda^k(f)_{\underline{i},\underline{j}} =$ coeff of $w_{i_1} \wedge \dots \wedge w_{i_k}$ in $\Lambda^k(f)(v_{j_1} \wedge \dots \wedge v_{j_k})$

Recall $w_{\sigma(i_1)} \wedge \dots \wedge w_{\sigma(i_k)} = \mathrm{sign}(\sigma)\, w_{i_1} \wedge \dots \wedge w_{i_k} \quad \forall \sigma \in S_k$.

So coeff of $w_{i_1} \wedge \dots w_{i_k}$ in $\left(\sum_{i=1}^{m} a_{ij_1} w_i\right) \wedge \dots \wedge \left(\sum_{i=1}^{m} a_{ij_k} w_i\right)$ (*)

is $\sum_{\sigma \in S_k} \mathrm{sign}(\sigma)\, a_{i_{\sigma(1)} j_1}\, a_{i_{\sigma(2)} j_2} \cdots a_{i_{\sigma(k)} j_k}$

In particular: If $k = n = m$ write $\det(f) = \Delta^{1,\dots,n}_{1,\dots,n}(f)$

This recovers the permutation formula for determinants.

$$\det(A) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma)\, a_{\sigma(1)1}\, a_{\sigma(2)2} \cdots a_{\sigma(n)n}.$$

$$\underset{\tau = \sigma^{-1}}{\overset{\downarrow}{=}} \sum_{\tau \in S_n} \mathrm{sign}(\tau)\, a_{1\tau(1)}\, a_{2\tau(2)} \cdots a_{n\tau(n)}$$

Consequence ① $\det(A) = \det(A^T) \qquad A \in \mathrm{Mat}_{n \times n}(\mathbb{K})$

Consequence ② Row-expansion formula for $\det(A)$

Pf/ Fix i^{th} Row of A

$$\det(A) = \sum_{j=1}^{n} a_{ij} \underbrace{\sum_{\sigma:\, \sigma(j)=i} \operatorname{sgn}(\sigma)\, a_{\sigma(1),1} \cdots \widehat{a_{\sigma(j),j}} \cdots a_{\sigma(n),n}}_{(**)}$$

restrict σ to $\tilde{\sigma} = \{1,\ldots,\hat{j},\ldots n\} \xrightarrow[b_{ij}]{} \{1,\ldots,\hat{i},\ldots,n\}$ so

$$\operatorname{sign}(\sigma) = (-1)^{i+j} \operatorname{sign}(\tilde{\sigma}).$$

so $(**) = \det\left(A^{(i,j)}\right)(-1)^{i+j}$

$$\Rightarrow \det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A^{(i,j)}. \qquad \square$$

Obs: Using ① & ② we get column expansion formula as well.

Consequence ③ If A has 2 equal rows (resp 2 equal cols) then $\det(A)=0$.

Pf/ $\Lambda^n(f)(v_1 \wedge \cdots \wedge v_n) = 0$ in $\Lambda^n(\mathbb{K}^n) \simeq \mathbb{K}$ by $(*)$ $\square$

Def (Cofactor matrix of A)

$$\operatorname{Cof}(A)_{i,j} = (-1)^{i+j} \det A^{(i,j)} \qquad 1 \leq i,j \leq n.$$

Consequence ④ $(\operatorname{Cof} A)^T A = \det(A)\, I_n = A(\operatorname{Cof} A)^T$

Pf/ $$\left((\operatorname{Cof} A)^T A\right)_{ij} = \sum_{l=1}^{n} (\operatorname{Cof} A)^T_{il}\, a_{lj}$$

$$= \sum_{l=1}^{n} (-1)^{i+l} \det\left(A^{(l,i)}\right) a_{lj}$$

- If $i=j$ This is j^{th} column expansion of $\det(A)$.
- If $i\neq j$ ______________ $\det(A')$ where

A' is the matrix obtained from A by replacing i^{th} col of A by the j^{th} col of A. By Consequence ③, $\det(A')=0$.

So $(\operatorname{Cof} A)^T A = \det A \, I_n$.

- $A(\operatorname{Cof} A)^T = ((\operatorname{Cof} A)A^T)^T = ((\operatorname{Cof} A^T)^T A^T)^T = (\det A^T I_n)^T$
$= \det A \, I_n$.

since $(\operatorname{Cof} A^T)_{ij} = (-1)^{i+j}\det(A^T)^{(i,j)} = (-1)^{i+j}\det A^{(j,i)} = (\operatorname{Cof} A)_{j,i}$. $\square$

<u>Consequence</u> ④ determinant is multiplicative. (see HW12)

<u>Proof</u>: Write $V\xrightarrow{f}W$ & $W\xrightarrow{g}U$ $\dim V=\dim W=\dim U=n$

$B=\{v_1,\dots v_n\}$, $B_2=\{w_1,\dots,w_n\}$, $B_3=\{u_1,\dots,u_n\}$ bases for V, W & U.

Then: $\Lambda^n(g\circ f)$:

$$\begin{tikzcd}
\Lambda^n(V) \arrow[r, "\Lambda^n(f)"] \arrow[d, phantom, "\shortparallel"] & \Lambda^n(W) \arrow[r, "\Lambda^n(g)"] \arrow[d, phantom, "\shortparallel"] & \Lambda^n(U) \arrow[d, phantom, "\shortparallel"] \\
Sp(v_1\wedge\dots\wedge v_n) \arrow[d, phantom, "\cong"] & Sp(w_1\wedge\dots\wedge w_n) \arrow[d, phantom, "\cong"] & Sp(u_1\wedge\dots\wedge u_n) \arrow[d, phantom, "\cong"] \\
\mathbb{K} \arrow[r, dashed, "\det(f)\cdot"] \arrow[rr, dashed, bend right, "\det(g\circ f)\cdot"'] & \mathbb{K} \arrow[r, dashed, "\det(g)"] \arrow[d, phantom, "\circlearrowleft"] & \mathbb{K}
\end{tikzcd}$$

So $\det(g\circ f) = \det(g)\cdot\det(f)$ gives the linear map $\mathbb{K}\longrightarrow\mathbb{K}$ on the bottom row. $\square$

§2 Permanents

Again fix V, W with $\dim V=n$, $\dim W=m$ & $V\xrightarrow{f}W$ linear.

Q: What happens if we do this for $S^k(V)\xrightarrow{S^k(f)}S^k(W)$?

A We get permanents! (HW12)

Def: $\mathrm{Perm}(F)_{\underline{i},\underline{j}} =$ coeff of $w_{i_1}\cdots w_{i_k}$ in $S^k(F)(v_{j_1}\cdots v_{j_k})$

$i_1<\cdots<i_k \quad j_1<\cdots<j_k$

⚠ We are NOT allowed to have repetitions, so we are not capturing all the coefficients of $S^k(F)$. Can include this by repeating columns.

In particular, for $n=m=k$, we have

$$\mathrm{Perm}(A) = \sum_{\sigma\in S_n} a_{\sigma(1)1}\cdots a_{\sigma(n)n}$$

⚠ It is no longer true that matrices with repeated rows have permanent $=0$. This makes it very hard to compute permanents! In particular, there are NO good algorithms for computing permanents. (Best results are due to A. Barvinok)

§3. Gaussian Decomposition:

Fix $X\in GL_n(\mathbb{K})$

Def. (HW12) We say X admits a Gaussian Decomposition if

$$X = X^- X^0 X^+$$

where $X^0 =$ diagonal matrix

$$X^- = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$X^+ = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$$

Theorem 1: Gaussian decompositions are unique.

Pf/ $X^- X^0 X^+ = Y^- Y^0 Y^+$

$$(Y^-)^{-1}X^- \quad X^0 \, (X^+)(Y^+)^{-1} = Y^0 \qquad \text{Gaussian decomp of } Y^0$$

since $(Y^-)^{-1}X^- = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$(X^+)(Y^+)^{-1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$

So it's enough to show it for diagonal matrices.

__Claim:__ $X^- X^0 X^+ = Y^0$ diagonal $\Longrightarrow X^- = X^+ = I_n$

Pf/ $\underbrace{X^0 X^+}_{\text{upper } \Delta} = \underbrace{(X^-) Y^0}_{\text{lower } \Delta}$ so both are diagonal (& invertible!)

$X^0 = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \Longrightarrow X^0 X^+ = \begin{pmatrix} x_1 & * & \cdots & * \\ & x_2 & & \vdots \\ & & \ddots & * \\ 0 & & & x_n \end{pmatrix} = \begin{pmatrix} y_1 & & & 0 \\ * & \ddots & & \\ \vdots & & \ddots & \\ * & \cdots & * & y_n \end{pmatrix}$

$Y^0 = \begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_n \end{pmatrix}$ forces $x_1 = y_1, \; x_2 = y_2, \; \ldots, \; x_n = y_n$

$X^+ = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ & 1 & & \\ & & \ddots & a_{n-1,n} \\ & & & 1 \end{pmatrix} \Longrightarrow \underset{j>i}{0 = (X^0 X^+)_{ij}} = \sum_{k=1}^{n} (X^0)_{ik}(X^+)_{kj}$

$$= (X^0)_{ii}(X^+)_{ij}$$
$$= \underset{\substack{\#\\0}}{x_i}\,(X^+)_{ij}$$

forces $(X^+)_{ij} = 0 \quad \forall\, j > i$ so $X^+ = I_n$.

Similarly: $X^- = I_n$ so $X^0 = Y^0$. ∎

__Theorem 2:__ X admits G.D $\Longleftrightarrow \Delta^{1,\ldots,i}_{1,\ldots,i}(X) \neq 0 \quad \forall\, i = 1, \ldots, n$

Pf/ $(\Longrightarrow)$ $\Delta^{1,\ldots,k}_{1,\ldots,k}(X) = \Delta^{1,\ldots,k}_{1,\ldots,k}(X^- X^0 X^+)$

$$= \Delta^{1,\ldots,k}_{1,\ldots,k}\left(\left(\begin{array}{c|c} \begin{matrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{matrix} & * \\ \hline 0 & \begin{matrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{matrix} \end{array}\right) \left(\begin{array}{c|c} \begin{matrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_k \end{matrix} & 0 \\ \hline 0 & \begin{matrix} d_{k+1} & & 0 \\ & \ddots & \\ 0 & & d_n \end{matrix} \end{array}\right) \left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{matrix} & 0 \\ \hline * & \begin{matrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{matrix} \end{array}\right) \right)$$

$$= \det\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_k \end{pmatrix} = d_1 \cdots d_k$$

So $\det(X) = d_1 \cdots d_n \neq 0 \iff$ all $d_i \neq 0$.

$\Rightarrow \Delta^{1,\dots,k}_{1,\dots,k}(\Delta) = d_1 \cdots d_k$ nonzero $\forall k$.

Moreover: $d_1 = \Delta^1_1(X)$

$d_1 d_2 = \Delta^{1,2}_{1,2}(\Delta) \rightsquigarrow d_2 = \dfrac{\Delta^{1,2}_{1,2}(X)}{\Delta^1_1(X)}$

In general $d_k = \dfrac{\Delta^{1,\dots,k}_{1,\dots,k}(X)}{\Delta^{1,\dots,k-1}_{1,\dots,k-1}(X)}$.

So X_0 is uniquely determined by the principal minors of X.

($\Leftarrow$) Explicitly, write $(X^0)_{kk} = \begin{cases} \Delta^1_1(X) & k=1 \\ \dfrac{\Delta^{1,\dots,k}_{1,\dots,k}(X)}{\Delta^{1,\dots,k-1}_{1,\dots,k-1}(X)} & k>1 \end{cases}$

$(X^-)_{ji} = \dfrac{\Delta^{1,\dots,i-1,j}_{1,\dots,i}(X)}{\Delta^{1,\dots,i}_{1,\dots,i}(X)}$, $(X^+)_{ji} = \dfrac{\Delta^{1,\dots,i}_{1,\dots,i-1,j}(X)}{\Delta^{1,\dots,i}_{1,\dots,i}(X)}$

and check $X = X^- X^0 X^+$ (by induction on n) $\square$

Obs: The construction is also true for $GL_n(R)$ where R is not necessarily a commutative ring (define dets via column expansion).

Obs 2: Non vanishing minors is an open condition in $\mathrm{Mat}_{n\times n}(\mathbb{K})$

We have a dense open set $\mathcal{U} \subseteq \mathrm{Mat}_{n\times n}(\mathbb{K})$ which we parameterize as

(Big Bruhat cell) $\mathcal{U} \cong \underset{\text{(entries in } X^-)}{\mathbb{K}^{\frac{n(n-1)}{2}}} \times \underset{\text{entries in } X^0}{(\mathbb{K}^*)^n} \times \underset{\text{entries in } X^+}{\mathbb{K}^{\frac{n(n-1)}{2}}}$

Can prove statements on $\mathrm{Mat}_{n\times n}(\mathbb{K})$ by restricting to $\mathcal{U}$.