Lecture 40: Bilinear forms, Sylvester's Theorem

§1. Bilinear forms:

Fix V_1, V_2, W $\mathbb{K}$-vector spaces

Recall: $\mathrm{Bil}_{\mathbb{K}}(V_1 \times V_2, W) = \{ f : V_1 \times V_2 \longrightarrow W \text{ bilinear} \}$

$f(v_1, -) \in \mathrm{Hom}_{\mathbb{K}}(V_2, W)$, $f(-, v_2) \in \mathrm{Hom}_{\mathbb{K}}(V_1, W)$

$\| $

$\mathrm{Hom}_{\mathbb{K}}(V_1 \otimes V_2, W)$

Def: A bilinear form on $V_1 \times V_2$ is an element of $\mathrm{Bil}(V_1 \times V_2, \mathbb{K})$

Def A bilinear form f on $V_1 \times V_2$ is non-degenerate if

$$V_1 \hookrightarrow V_2^* \quad \& \quad V_2 \hookrightarrow V_1^*$$
$$v_1 \longmapsto f(v_1, -) \qquad\quad v_2 \longmapsto f(-, v_2).$$

So if V_1 & V_2 have finite-dimensions, we get $V_1 \simeq V_1^*$, $V_2 \simeq V_2^*$ & so if f is nondegenerate we get $\boxed{\dim V_1 = \dim V_2}$

Motivation: Poincaré Duality. Fix X smooth compact manifold of $\dim = n$

(V1) $H_k(X) \otimes H^k(X) \xrightarrow{\int} \mathbb{R}$ is non-deg. $\forall\, 0 \leq k \leq n$

$$\sum_{i=1}^{m} a_i S_i \otimes \sum_{j=1}^{\ell} b_j \Psi_j \longmapsto \sum_{i,j} a_i b_j \int_{S_i} \Psi_i \qquad a_i, b_j \in \mathbb{R}$$

S_i = k-cell (simplicial) : $\Delta_k \hookrightarrow S_i \subseteq X$

Ψ_i : k-form on X (exterior) $\qquad \displaystyle\int_{S_i} \Psi_i = \int_{\Delta_k} S^* \Psi_i$ (Riemann integral)

(V2) $H^{n-k}(X) \otimes H^k(X) \longrightarrow \mathbb{R} \qquad \forall\, 0 \leq k \leq n$

$$\Psi \otimes \xi \longmapsto \int_X \Psi \wedge \xi$$

Poincaré duality: $H_k(X) \underset{(V1)}{\simeq} (H^k(X))^* \underset{(V2)}{\simeq} H^{n-k}(X) \quad \forall\, 0 \leq k \leq n$

- Assume $\dim V_1 = \dim V_2 = n$ & pick $B_1 = \{v_1, \ldots, v_n\}$ basis for V_1, $B_2 = \{w_1, \ldots, w_n\}$ ——— V_2

Write $f \in \mathrm{Bil}(V_1 \times V_2, \mathbb{K})$ via $\langle v_i, w_j \rangle = f(v_i, w_j)$

& build an $n \times n$ matrix $\boxed{Q}$ with $Q_{i,j} = \langle v_i, w_j \rangle \ \forall i,j$

Prop: f is non-degenerate if & only if Q is invertible.

Pf/ ($\Rightarrow$)

$$V_1 \xrightarrow{\varphi_1} V_2^*$$
$$v \longmapsto \{w \mapsto f(v,w)\}$$

So $\varphi_1(v_j)(w_i) = f(v_j, w_i) = Q_{ji}$

This means $\varphi_1(v_j) = \sum_{i=1}^{n} Q_{ji} w_i^*$

$$[\varphi_1]_{B_1, B_2^*} = Q^T$$

Since φ_1 is inj & $\dim V_1 = \dim V_2^* = n < \infty$, we conclude φ_1 is an iso. Thus, we conclude Q^T (and hence Q) is invertible.

($\Leftarrow$) Q is invertible so Q^T are invertible. Since $[\varphi_1]_{B_1, B_2^*} = Q^T$, we conclude φ_1 is injective.

Similarly $V_2 \xrightarrow{\varphi_2} V_1^*$

$$w_j \longmapsto (v_i \mapsto f(v_i, w_j) = Q_{ij})$$

So $\varphi_2(w_j) = \sum_{i=1}^{n} Q_{ij} v_i^*$ & so $[\varphi_2]_{B_2, B_1^*} = Q$.

Conclude: φ_2 is an iso, so it is injective.

Since both φ_1 & φ_2 are injective, we conclude that f is non-degenerate. $\square$

§2 Symmetric, skew-symmetric & alternating forms:

Def: We say f in $\mathrm{Bil}(V,V,\mathbb{K})$ is

1. symmetric if $f(v,w) = f(w,v)$ $\forall v,w \in V$
2. skew-symmetric if $f(v,w) = -f(w,v)$ $\forall v,w \in V$
3. alternating if $f(v,v) = 0$ $\forall v \in V$.

same unless char $\mathbb{K} = 2$

Ex 1 Dot Product on $V = \mathbb{R}^n$. is symmetric.

In general $f(x,y) = x^T Q y$ with $Q = Q^T$ gives a symmetric bilinear form on $\mathbb{K}^n$.

Ex 2 $V = \mathbb{K}$ & $f(x,y) = xy$ is symmetric not alternating bilinear form. Skew symmetric when char $\mathbb{K} = 2$

Ex 3 $f((x,y),(x',y')) = xy' - x'y = \det\begin{pmatrix} x & x' \\ y & y' \end{pmatrix}$ is skew-symmetric & alternating bilinear form on $\mathbb{R}^2$.

Ex 4 Pick $u \in \mathbb{R}^3$, $f_u(v,w) = u \cdot (v \times w)$ is alternating, skew-sym. form on $\mathbb{R}^3$.

Ex 5 $\dim V = n < \infty$ & $W = \mathrm{End}_{\mathbb{K}}(V) = \mathrm{Mat}_{n\times n}(\mathbb{K})$

$f \in \mathrm{Bil}(W \times W, \mathbb{K})$ via $f(A,A') = \mathrm{Tr}(AA')$ Trace form symmetric by construction.

Ex 6 $V = C[0,1] = \{h : [0,1] \longrightarrow \mathbb{R}$ cont$\}$. inf-dim'l $\mathbb{R}$-v.s.

- $F : V \times V \longrightarrow \mathbb{R}$ via $F(g,h) = \int_0^1 g(x)\, h(x)\, dx$

F is a symmetric bilinear form.

- Pick $k : [0,1]^2 \longrightarrow \mathbb{R}$ continuous:

$F_k : V\times V \longrightarrow \mathbb{R}$ via $F_k(g,h) = \iint_{[0,1]^2} g(x)\, h(y)\, k(x,y)\, dx$

- F_k bilinear but not symmetric. (unless k is).

Lemma 1: $f \in \mathrm{Bil}(\mathbb{K}^n, \mathbb{K}^n, \mathbb{K})$ with associated matrix Q. Then

1. f is symmetric if & only if $Q = Q^T$ (symmetric matrix)
2. f is skew-symmetric if & only if $Q^T = -Q$ (skew-sym matrix)
3. f is alternating if & only if $Q^T = -Q$ & $Q_{ii} = 0$ $\forall i$.

(($\Leftarrow$) for 3 uses $f(v,v) = 0$ $\forall v \Rightarrow f(v,w) = -f(w,v)$ $\forall v,w$ since $f(v+w, v+w) = 0$.)

Proposition: $\mathrm{Bil}(V\times V, \mathbb{K}) \overset{\varphi}{\simeq} \mathrm{Bil}^{\mathrm{sym}}(V\times V, \mathbb{K}) \oplus \mathrm{Bil}^{\text{skew-sym}}(V\times V, \mathbb{K})$
(for char $\mathbb{K} \neq 2$)

Pf/ $\varphi(f) = f_1 + f_2$ with $f_1(v,w) = \dfrac{f(v,w) + f(w,v)}{2}$

$f_2(v,w) = \dfrac{f(v,w) - f(w,v)}{2}$ $\square$

Alternative: $(V\otimes V)^* \underset{\text{fin dim.}}{\simeq} V^*\otimes V^* \simeq S^2(V^*) \oplus \Lambda^2(V^*)$
(if $\dim V < \infty$)

Lemma 2: If char $\mathbb{K} \neq 2$ then $f \in \mathrm{Bil}^{\mathrm{sym}}(V\times V, \mathbb{K})$ is completely determined by the values $f(v,v)$ $\forall v \in V$.

Pf/ Say we want to determine $f(v,w)$, then

$$\begin{aligned} f(v+w, v+w) &= f(v, v+w) + f(w, v+w) \\ &= f(v,v) + f(v,w) + f(w,v) + f(w,w) \\ &= f(v,v) + 2f(v,w) + f(w,w) \end{aligned}$$

So $f(v,w) = \dfrac{f(v+w, v+w) - f(v,v) - f(w,w)}{2}$ $\square$

Q: How to work with degenerate symm. forms in $Bil(V,V,\mathbb{K})$?

A: Given $f \in Bil^{symm}(V,V,\mathbb{K})$, & $v \in V$, set

$v^\perp = \{w \in V \mid f(v,w) = 0\}$.

$Rad(f) = \{v \mid f(v,-) = 0 \in V^*\} \subseteq V$ subspace.

If $V' = V/Rad(f)$ View $V = \underset{\underset{V'}{\wr\wr}}{\boxed{W}} \oplus Rad(f)$ (via a section)

Claim: $f|_{W\times W}$ is non-deg

Pf/ The matrix for f has the form
$$\begin{array}{ccc} & W & Rad(f) \\ & \left(\begin{array}{c|c} Q & 0 \\ \hline 0 & \underbrace{0}_{v\in Rad(f)} \end{array}\right) & \begin{array}{c} W \\ Rad(f) \end{array} \end{array}$$
by symmetry

If $w \in \ker(\varphi_1 : W \longrightarrow W^*)$
$\qquad\qquad\qquad\quad w' \longmapsto f(w,w')$

then $f(w,w') = 0 \quad \forall w' \in W$.
But $f(w,v) = 0 \quad \forall v \in Rad(f)$
$\Big\}$ $f(w,v) = 0 \ \forall v \in V$ so $w \in Rad(f)$

Conclude $w \in W \cap Rad(f) = 0$, so $\varphi_1 : W \hookrightarrow W^*$.

Since f is symmetric $\varphi_2 : W \hookrightarrow W^*$ as well. □

Obs: Same idea works for skew-symmetric forms (we have $v^\perp$)

§3 Sylvester's Theorem:

GOAL Classify symmetric non-deg bilinear forms on $V \cong \mathbb{R}^n$ via invariants (rank & signature)

STEP 1: Degenerate vs non-degenerate

① 1st invariant = rank of f $= rk([f]) = \dim V - \dim Rad(f)$

STEP 2: Classify non-deg symm forms = Sylvester's Thm

(2) 2^{nd} invariant = signature of f

Sylvester's Theorem: Fix $f: V \times V \longrightarrow \mathbb{K}$ non degenerate symmetric bilinear form on V. Assume that $\operatorname{char} \mathbb{K} \neq 2$ & $\dim V = n < \infty$

Then $\exists$ basis $\mathcal{B} = \{\varepsilon_1, \ldots, \varepsilon_n\}$ of V s.t. $f(\varepsilon_i, \varepsilon_j) = \pm \delta_{ij}$

So, after reordering, the matrix for f in the basis $\mathcal{B}$ becomes:

$$Q = (f(\varepsilon_i, \varepsilon_j))_{ij} = \left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{matrix} \end{array}\right]$$

Moreover, the # of >0 is independent of the bases.

Def Signature(f) = # 1's (For degenerate forms = (# 1's, # -1's))

Idea: Find an "orthonormal" basis, up to sign using Gram-Schmidt to build an orthogonal basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ from an input basis $\{v_1, \ldots, v_n\}$ for V.

($\varepsilon_1 = v_1$, $\varepsilon_2 = v_2 - \frac{\langle \varepsilon_1, v_2 \rangle}{\langle \varepsilon_1, \varepsilon_1 \rangle} \varepsilon_1$, $\varepsilon_3 = v_3 - \frac{\langle \varepsilon_1, v_3 \rangle}{\langle \varepsilon_1, \varepsilon_1 \rangle} \varepsilon_1 - \frac{\langle \varepsilon_2, v_3 \rangle}{\langle \varepsilon_2, \varepsilon_2 \rangle} \varepsilon_2$, etc.)

But we can't do this directly since $v \neq 0 \not\Rightarrow f(v,v) \neq 0$.

Next Time: we'll see how to by-pass this.