Lecture 41: Classification of symmetric & skew-symm forms on $\mathbb{K}^n$, char $\mathbb{K} \neq 2$

Last Time $f: V \times V \longrightarrow \mathbb{K}$ bilinear form is

1) non-degenerate if $\varphi_1 : V \longrightarrow V^*$, $v \longmapsto (w \mapsto f(v,w))$ & $\varphi_2 : V \longrightarrow V^*$, $v \longmapsto (w \mapsto f(w,v))$ are injective

2) symmetric if $f(v,w) = f(w,v)$ $\forall v, w$

3) skew-symmetric if $f(v,w) = -f(w,v)$ $\forall v,w$ $(\Rightarrow 2f(v,v) = 0 \ \forall v)$

Def: $\mathrm{Rad}(f) = \mathrm{Ker}\,\varphi_1 \subset V$ subspace.

Prop: If $\dim V < \infty$, & f is either symmetric or skew-symmetric, then:

$$\mathrm{Rad}(f) = 0 \iff f \text{ is non-degenerate}$$

- From now on, assume $\mathrm{char}(\mathbb{K}) \neq 2$.

Fix $f \in \mathrm{Bil}(\mathbb{K}^n \times \mathbb{K}^n, \mathbb{K})$ symmetric or skew-symmetric. Then, we can pick a basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ for $\mathbb{K}^n$ where $\mathrm{Sp}(\mathcal{B}_2) = \mathrm{Rad}(f)$, $W := \mathrm{Sp}(\mathcal{B}_1)$ & $\tilde{f} = f|_{W \times W} : W \times W \longrightarrow \mathbb{K}$ is non-degenerate

$$[f]_{\mathcal{B}\mathcal{B}} = \begin{array}{c} \\ m \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{cc} m & \end{array} \\ \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}$$

Q in $GL_m(\mathbb{K})$. $Q = [\tilde{f}]_{\mathcal{B}_1\mathcal{B}_1}$

$\mathrm{Rank}(f) = \mathrm{rk}\, Q = \mathrm{Rank}(\tilde{f})$

(1) f is symmetric $\iff$ $\tilde{f}$ is symmetric $\iff$ $Q = Q^T$

(2) f is skew-symmetric $\iff$ $\tilde{f}$ is skew-symmetric $\iff$ $Q = -Q^T$.

For classification purposes, it's enough to look at non-degenerate forms.

§1. Symmetric, non-degenerate bilinear forms:

Assume $\mathrm{char}(\mathbb{K}) \neq 2$

Proposition: Fix $f: \mathbb{K}^n \times \mathbb{K}^n \longrightarrow \mathbb{K}$ a non-degenerate symmetric bilinear form. Then $\exists$ basis $\mathcal{B} = \{v_1, \dots, v_n\}$ with $f(v_i, v_j) = 0$ for all i,j with $i \neq j$ & $f(v_i, v_i) \neq 0$. So $[f]_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$ & $* \neq 0$.

In particular, we have

1. If $\mathbb{K}=\overline{\mathbb{K}}$, we can choose $\{v_1,\ldots,v_n\}$ with $f(v_i,v_j)=\delta_{ij}$
2. [Sylvester's Theorem] If $\mathbb{K}=\mathbb{R}$, we can choose $\{v_1,\ldots,v_n\}$ with $f(v_i,v_j)=\pm\delta_{ij}$.

Proof of 1 & 2:

1. Given $f(v_i,v_i)=:\alpha_i\neq 0$ we let $a_i\in\mathbb{K}$ be a solution to $x^2-\alpha_i=0$ in $\mathbb{K}$. Take $\tilde{v}_i=\frac{v_i}{a_i}$ $\forall i$ & conclude $f(\tilde{v}_i,\tilde{v}_i)=\frac{1}{a_i^2}f(v_i,v_i)=\frac{\alpha_i}{a_i^2}=1$ $\forall i$

2. We proceed in a similar fashion, but pick a_i a solution to $x^2-|\alpha_i|=0$ in $\mathbb{R}$ ($a_i=\sqrt{|\alpha_i|}$). Then $\tilde{v}_i=\frac{v_i}{a_i}$ satisfies
$$f(\tilde{v}_i,\tilde{v}_i)=\frac{1}{a_i^2}f(v_i,v_i)=\frac{\alpha_i}{|\alpha_i|}=\pm 1$$
, as we wanted.

Proof of the 1st part of the Proposition:

We proceed by induction on n.

(1) Base case: $n=1$ $\quad f(a,b)=f(a\cdot 1,b\cdot 1)=ab\,f(1,1)$

Since f is non-degenerate, $f(1,1)\neq 0$ (otherwise, $f\equiv 0$ which is deg).

Then $v=1$ gives the desired basis for $\mathbb{K}$

(2) Inductive Step: The main issue we face is how to find v_1, since "$v\neq 0 \not\Rightarrow f(v,v)\neq 0$".

We need the following key lemma to bypass this when char $\mathbb{K}\neq 2$.

Lemma 1: Any symmetric bilinear form on a $\mathbb{K}$-vector space V with char $\mathbb{K}\neq 2$ is completely determined by $\{f(v,v) : v\in V\}$

Proof $f(v+w,v+w)=f(v,v+w)+f(w,v+w)=f(v,v)+f(v,w)$
$$+f(w,v)+f(w,w)\underset{f\text{ sym}}{=}f(v,v)+2f(v,w)+f(w,w)$$

$$\Rightarrow f(v,w) = \frac{f(v+w, v+w) - f(v,v) - f(w,w)}{2} \quad \forall v,w \in V.$$

$\square$

The induction will be a direct consequence of the following lemma. Indeed, take $\mathcal{B} = \{v_0\} \cup \{w_1, \ldots, w_{n-1}\}$ with $\{w_1, \ldots, w_{n-1}\}$ basis for $\langle v_0\rangle^\perp$ for $\tilde{f} = f|_{\langle v_0\rangle^\perp \times \langle v_0\rangle^\perp}$

Lemma 2: Assume f is symmetric, non-deg $\mathbb{K}$-bilinear form, on $\mathbb{K}^n$ and char $\mathbb{K} \neq 2$. Then:

1. $\exists\, v_0 \neq 0$ with $f(v_0, v_0) \neq 0$
2. $\mathbb{K}^n = Sp(v_0) \oplus \langle v_0\rangle^\perp$ where $\langle v_0\rangle^\perp = \{w \in \mathbb{K}^n : f(v_0,w) = 0\}$
3. $f|_{\langle v_0\rangle^\perp \times \langle v_0\rangle^\perp}$ is a non-degenerate symmetric bilinear form.

Pf/ 1. If $f(x,x) = 0$ $\forall x \in \mathbb{K}^n$, then $f(v,w) = 0$ $\forall v,w$ by Lemma 1, i.e. $f \equiv 0$. This cannot happen since f is nondegenerate. Thus, $\exists\, v_0 \in \mathbb{K}^n$ with $f(v_0, v_0) \neq 0$

2. Build $W = \langle v_0\rangle^\perp$. By 1. $v_0 \notin W$.

Since $W = \ker\left(\begin{array}{ccc} \mathbb{K}^n & \xrightarrow{\varphi_1(v_0)} & \mathbb{K} \\ w & \longmapsto & f(v_0,w) \end{array}\right)$ we know it is a subspace of $\mathbb{K}^n$.

- Claim: $W \cap Sp(v_0) = \{0\}$

Indeed, $f(v_0, \lambda v_0) = \lambda \underbrace{f(v_0,v_0)}_{\neq 0}$ so $\lambda v_0 \in W \iff \lambda = 0$.

- By the Rank-Nullity Theorem, $\dim W + \underbrace{rk(\varphi_1(v_0))}_{=1 \text{ since } \varphi_1(v_0) \neq 0 \text{ in } (\mathbb{K}^n)^*} = n$

So $\dim W = n-1$.

- $\dim(W + Sp(v_0)) \underset{\text{Claim}}{=} n$ so $\mathbb{K}^n = \langle v_0\rangle \oplus W$.

3. Need to show $\begin{array}{ccc} W & \xrightarrow{\tilde{\varphi}_1} & W^* \\ w & \longmapsto & (v \mapsto f(w,v)) \end{array}$ is injective. Symmetry will say $\tilde{\varphi}_2 : W \to W^*$ is injective as well.

• Pick $w \in \operatorname{Ker}(\varphi_1)$ so $f(w,v)=0 \quad \forall v \in W$

But $w \in \langle v_0\rangle^{\perp}$ so $f(w,\lambda v_0) = \lambda f(w,v_0) = 0 . \quad \forall \lambda \in \mathbb{K}.$

It follows that $f(w,-) = 0 \in (\mathbb{K}^n)^*$ by ②.

This implies $w \in \ker (V \xhookrightarrow{\varphi_1} V^*) = \{0\}$.

Conclusion: $\tilde{f} = f|_{W\times W}$ is a non-deg symm. bilinear form. $\square$

Sylvester's Theorem: For $\mathbb{K}=\mathbb{R}$, the number of #1's is independent of the choice of basis $\{v_1, \ldots v_n\}$ in the Proposition.

Proof Say we have 2 basis as in the Proposition

$\mathcal{B} = \{v_1, \ldots, v_p, v_{p+1}, \ldots v_n\}$ with $f(v_i,v_i) = \begin{cases} 1 & i \leq p \\ -1 & i > p \end{cases}$

$\mathcal{B}' = \{w_1, \ldots, w_q, w_{q+1}, \ldots, w_n\}$ — $f(w_i,w_i) = \begin{cases} 1 & i \leq q \\ -1 & i > q \end{cases}$

$f(v_i,v_j) = f(w_i,w_j) = 0 \quad \forall i \neq j$

• Assume $p<q$ & reach a contradiction. Symmetry yields $p=q$

We define a linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^{p+n-q}$

$$\xi \longmapsto \begin{bmatrix} f(v_1,\xi) \\ \vdots \\ f(v_p,\xi) \\ f(w_{q+1},\xi) \\ \vdots \\ f(w_n,\xi) \end{bmatrix}$$

Since $p+n-q<n$, the rank-nullity theorem yields: $\ker L \neq \{0\}$. Pick $\xi_0 \in \ker L \smallsetminus \{0\}$

Claim: $\xi_0 \in Sp(v_{p+1}, \ldots v_n) \cap Sp(w_1, \ldots, w_q)$

Indeed $\xi_0 = \sum_{i=1}^{n} a_i v_i \Rightarrow 0 = f(v_i,\xi_0) = a_i \underbrace{f(v_i,v_i)}_{=1} \quad \forall i \leq p$

so $\xi_0 \in Sp(v_{p+1}, \ldots v_n)$

$\xi_0 = \sum_{i=1}^{n} b_i w_i \Rightarrow 0 = f(w_i,\xi_0) = -b_i \underbrace{f(w_i,w_i)}_{\neq -1} \quad \forall i > q$, so $\xi_0 \in Sp(w_1, \ldots w_q)$

Now $f(\xi_0,\xi_0) = f\left(\sum_{i=p+1}^{n} a_i v_i, \sum_{i=p+1}^{n} a_i v_i\right)$

$$= \sum_{i,j=p+1}^{n} a_i a_j\, f(v_i,v_j) = \sum_{i=p+1} -a_i^2 \leq 0 \qquad (1)$$

$$f(\xi_0,\xi_0) = f\left(\sum_{i=1}^{q} b_i w_i, \sum_{i=1}^{q} b_i w_i\right)$$

$$= \sum_{i,j=1}^{q} b_i b_j\, f(w_i,w_j) = \sum_{i=1}^{q} b_i^2 \geq 0 \qquad (2)$$

But (1) & (2) gives $f(\xi_0,\xi_0)=0$ so $\sum_{i=1}^{q} b_i^2 = 0$. This gives $b_1 = \cdots = b_q = 0$, ie $\xi_0 = \sum_{i=1}^{q} 0\cdot w_i = 0$ Contra! $\square$

Consequence: Classification of quadratic forms q in $\mathbb{R}^n$ ($=$ homogeneous degree 2 polynomials in $\mathbb{R}[x_1,\ldots x_n]$)

After a linear change of coordinates, they become

$$x_1^2 + \cdots + x_s^2 - x_{s+1}^2 - \cdots - x_r^2$$

Pf/ $f(v,w) = \frac{1}{2}\left(q(v_1+w_1,\ldots,v_n+w_n) - q(\underline{v}) - q(\underline{w})\right)$

is a bilinear form $r = \text{rank}(f)$, $s = \text{signature}(f)$

Note: $q(\xi) = f(\xi,\xi)$

- Change of basis in $\mathbb{R}^n$: $\{e_1,\ldots,e_n\} \longrightarrow \{w_1,\ldots,w_r, \underbrace{w_{r+1},\ldots,w_n}_{\text{in } Rad(f)}\}$

Next, we view $W = Sp(w_1,\ldots w_r) = \mathbb{R}^r$ & consider

$$\tilde{f} = f|_{W\times W} : \mathbb{R}^r \times \mathbb{R}^r \longrightarrow \mathbb{R} \quad \text{symm, non-deg bilinear form}$$

with $s = $ signature of $(\tilde{f})$

By Sylvester's Theorem: $\exists$ basis $\{\varepsilon_1,\ldots,\varepsilon_r\}$ of $\mathbb{R}^r = W$ with

with $f(\varepsilon_i,\varepsilon_i) = \begin{cases} 1 & i\leq s \\ -1 & i>s \end{cases}$ & $f(\varepsilon_i,\varepsilon_j) = 0\ \forall i\neq j$.

• Change of basis 2 in W: $\{w_1, \dots w_r\} \longrightarrow \{\varepsilon_1 \dots \varepsilon_r\}$

Conclusion: $\{e_1, \dots, e_n\} \xleftrightarrow{\varphi} \{\varepsilon_1, \dots \varepsilon_r, w_{r+1}, \dots w_n\}$ (φ is a linear change of coordinates)

$$q \longmapsto \tilde{q}(x) = x_1^2 + \dots + x_s^2 - x_{s+1}^2 - \dots - x_r^2.$$

§2. Skew-symmetric, non-degenerate bilinear forms:

Assume $\operatorname{char}(\mathbb{K}) \neq 2$

• In what follows we present the analog of the above Proposition for skew-symm non-deg $\mathbb{K}$-bilinear forms on $V \simeq \mathbb{K}^n$. This result is used to build Darboux coordinates in real symplectic manifolds.

• We start discussing the dimension of V.

Lemma 3: If $\exists f \in \operatorname{Bil}^{\text{Skewsym}}_{\text{nondeg}}(V \times V, \mathbb{K})$ & $\dim V < \infty$ then $\dim V$ is even.

Proof: Pick $Q = [f]_{BB} \in \operatorname{Mat}_{n \times n}(\mathbb{K})$ when B is any basis for V.

Then $Q^T = -Q$ since f is skew-symm. & Q is invertible because f is nondegenerate.

Then $\det(Q) = \det(-Q^T) = (-1)^n \det Q^T = (-1)^n \det Q$

Since $\det Q \neq 0$ & $\operatorname{char} \mathbb{K} \neq 2$ we have $(-1)^n = 1$ so n is even. □

Theorem: Let $f: V \times V \longrightarrow \mathbb{K}$ be a non-deg skew-symmetric form on a finite-dimensional $\mathbb{K}$-vector space V, with $\operatorname{char} \mathbb{K} \neq 2$.

Then $\exists$ basis $B = \{\varepsilon_i, \eta_i\}_{i=1}^n$ for V s.t.

1. $f(\varepsilon_i, \eta_j) = \delta_{ij} = -f(\eta_j, \varepsilon_i)$ $\forall i, j$
2. $f(\varepsilon_i, \varepsilon_j) = 0 = f(\eta_i, \eta_j)$ $\forall i, j$.

$$[f]_{BB} = \begin{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{bmatrix}$$

$n = \dim V / 2$

(Alt: $[f]_{\varepsilon, \eta} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$)

Proof: By Lemma 3, $\dim V$ is even ($=2n$ for some $n \in \mathbb{Z}_{\geq 1}$)
We proceed by induction on n.

Base case: $n=1$ Pick any $\varepsilon_1 \in V \setminus \{0\}$ & choose $\eta_1' \in V$ st $f(\varepsilon_1, \eta_1') \neq 0$ (such η_1 exists since $f(\varepsilon_1, -) \neq 0$ in V^*)

Claim: $\{\varepsilon_1, \eta_1'\}$ is li.

Otherwise, $\eta_1' = \alpha \varepsilon_1$ (since $\varepsilon_1 \neq 0$), so $f(\varepsilon_1, \eta_1') = \alpha \underbrace{f(\varepsilon_1, \varepsilon_1)}_{=0} = 0$ Contr!

Next, we rescale η_1' to $\eta_1 = \dfrac{\eta_1'}{f(\varepsilon_1, \eta_1')}$ so $f(\varepsilon_1, \eta_1) = 1$.
. $\dim V = 2$ so we are done.

Inductive Step: Pick $\{\varepsilon_1, \eta_1\}$ as in the base case.
Next, consider $W = \{v \in V : f(\varepsilon_1, v) = f(\eta_1, v) = 0\}$
By Lemma 4 (below), we have:

1. $\dim W = 2n-2 = 2(n-1)$
2. $W \oplus Sp(\varepsilon_1, \eta_1) = V$
3. $f|_{W \times W}$ is non-deg & skew-sym.

So by the IH, W has a basis $\{\varepsilon_i, \eta_i\}_{i=2}^{n}$ satisfying the conditions in the statement. The conditions
$f(\varepsilon_1, \varepsilon_j) = f(\eta_1, \eta_j) = 0 \quad \forall j \neq 1$ follow from the def of W.
$f(\varepsilon_1, \eta_j) = f(\eta_1, \varepsilon_j) = 0 \quad \forall j \neq 1$ ——— $\square$

Lemma 4: Let f be a non-deg skew-sym form on $V \simeq \mathbb{K}^{2n}$ with char$\mathbb{K} \neq 2$. Assume V admits 2 vectors li ε_1, η_1 with $f(\varepsilon_1, \eta_1) = 1$
Consider $W = \langle \varepsilon_1, \eta_1 \rangle^{\perp} = \{w \in V : f(\varepsilon_1, w) = f(\eta_1, w) = 0\}$.
Then: $V = Sp(\varepsilon_1, \eta_1) \oplus W$ & $f|_{W \times W}$ is non-deg.

Proof: We start by showing the sum is direct

Claim 1: $W \cap Sp(\varepsilon_1, \eta_1) = \{0\}$.

Pf/ Pick $\omega = \alpha\varepsilon_1 + \beta\eta_1 \in W$. We show $\alpha = \beta = 0$ as follows.

(1) $0 = f(\varepsilon_1, \alpha\varepsilon_1 + \beta\eta_1) = \alpha \underbrace{f(\varepsilon_1, \varepsilon_1)}_{=0} + \beta \underbrace{f(\varepsilon_1, \eta_1)}_{=1} = \beta \Rightarrow \beta = 0.$

(2) $0 = f(\eta_1, \alpha\varepsilon_1) = \alpha f(\eta_1, \varepsilon_1) = -\alpha \Rightarrow \alpha = 0.$

Claim 2: $\dim W = \dim V - 2$

Pf/ $\varphi : V \longrightarrow \mathbb{K}^2$ is linear & $W = \operatorname{Ker}\varphi$

$v \longmapsto \begin{bmatrix} f(\varepsilon_1, v) \\ f(\eta_1, v) \end{bmatrix}$

$\varphi(-\varepsilon_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\varphi(\eta_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so $\operatorname{Im}\varphi = \mathbb{K}^2$

By Rank-Nullity Theorem, $\dim W = \dim V - 2$.

Claim 3: $f|_{W \times W}$ is non-deg.

Pf/ $\varphi_1 : W \longrightarrow W^*$, $v \longmapsto f(v, -)$ $\qquad$ $\varphi_2 = -\varphi_1 : W \longrightarrow W^*$, $v \longmapsto f(-, v)$

Pick $v \in \ker(\varphi_1)$ so $f(v, \omega) = 0$ $\forall \omega \in W$.

Now $v \in W$ so $f(v, \varepsilon_1) = f(v, \eta_1) = 0$.

Conclude $f(v, -) = 0$ in V^* $\Longrightarrow v = 0$ since f is non-deg. $\square$