Math 6501 - Enumerative Combinatorics I – Homework 1

Due at 3:00pm on Friday September 6th, 2019

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams, but individual solutions **must** be submitted for grading and credit. If you work in teams, please indicate the name of your collaborator(s) for each problem.

Problem 1. Let V be a finite set. A *graph* $G = (V, E)$ is a collection E of 2-element subsets of V. The elements of V are called *vertices* and the pairs in E are called *edges*.

(i) List (or draw) the graphs with vertex set $[4] = \{1, 2, 3, 4\}$ if we consider the elements of V to be distinct (e.g., by labeling them).

(ii) Repeat the above question but now assume the elements of V are *indistinguishable*, i.e. the graphs $([4], \{(1, 2)\})$ and $([4], \{(2, 3)\})$ are identified.

(iii) Give a general procedure to count the graphs on V (with $|V| = n$) in the labeled and unlabeled cases.

Problem 2. Let $S_1, \ldots, S_m$ be pairwise disjoint sets with $|S_j| = a_j \in \mathbb{Z}_{>0}$.

(i) Show that the number of subsets of $S_1 \cup \ldots \cup S_m$ containing at most one element from each S_j is the product $(a_1 + 1) \cdots (a_m + 1)$.

(ii) Apply the previous item to the following number-theoretic problem. Let $n = p_1^{a_1} \ldots p_m^{a_m}$ be the prime decomposition of n. Then, the number of divisors of n equals $(a_1 + 1) \cdots (a_m + 1)$. Conclude that n is a perfect square precisely when this number is odd.

Problem 3. Define *Euler's φ-function* as $\varphi(n) := \#\{k : 1 \le k \le n, k \text{ relatively prime to } n\}$. Show that $\sum_{d|n} \varphi(d) = n$ by decomposing $[n] := \{1, 2, \ldots, n\}$ into disjoint sets.

Problem 4. Let $f(n, k)$ be the number of k-subsets of $[n]$ not containining a pair of consecutive integers.

(i) Show that $f(n, k) = \binom{n-k+1}{k}$.

(ii) Show that $\sum_{k=0}^{n} f(n, k) = F_{n+2}$, the $(n + 2)$th. Fibonacci number.

Problem 5. Show that the sum of the right-left diagonals in Pascal's triangle (viewed in matrix form) ending at $(n, 0)$ is the $(n + 1)$th. Fibonacci number F_{n+1}.

Problem 6. Let $L(m, n)$ be the number of lattice paths from $(0, 0)$ to (m, n) using steps $(1, 0)$ and $(0, 1)$.

(i) Show that $L(m, n)$ satisfies the following recursion: $L(m, n) = L(m - 1, n) + L(m, n - 1)$, where $L(m, n) = 0$ if m or n are negative. Conclude that $L(m, n) = \binom{m+n}{n}$.

(ii) Use item (i) to give a combinatorial proof of the following variant of the **Vandermonde Identity**:

$$\sum_{k=0}^{n} \binom{s+k}{k} \binom{n-k}{m} = \binom{s+n+1}{s+m+1} \quad \text{for } s, m, n \in \mathbb{Z}_{\ge 0}.$$

Problem 7. Consider the lattice $\mathbb{Z}^2$ and let m, n be two non-negative integers. A *Delannoy path* from $(0, 0)$ to (m, n) is a lattice path that uses steps $(1, 0)$, $(0, 1)$ and $(1, 1)$. Let $D_{m,n}$ be the number of such paths, which we call *Delannoy number*.

(i) Draw all five paths corresponding to $m = 2$ and $n = 1$ (i.e., $D_{2,1} = 5$).

(ii) Prove that $D_{m,n} = \sum_k \binom{m}{k}\binom{n+k}{m}$ (note that the sum is finite!).
(*Hint:* Classify the paths by the number of diagonal steps taken.)