35/35

MATH 6501 - HOMEWORK 1

QIAN CHAO

Solution 1. (i) Let $a = (1,2)$, $b = (1,3)$, $c = (1,4)$, $d = (2,3)$, $e = (2,4)$, $f = (3,4)$. Then we have in total $2^6 = 64$ different graphs, which are $([4], S)$ where S is any subset of $X = \{a,b,c,d,e,f\}$. ✓

(ii) There are 11 of them, using the same notation in (a), they are: $([4], \phi)$, $([4], \{a\})$, $([4], \{a,b\})$, $([4], \{a,f\})$, $([4], \{a,b,c\})$, $([4], \{a,d,f\})$, $([4], \{a,b,d\})$, $([4], \{a,c,d,f\})$, $([4], \{a,b,c,d\})$, $([4], \{a,b,c,d,e\})$, $([4], \{a,b,c,d,e,f\})$. ✓

(iii) For labeled case, if there are n vertices, there are totally $\binom{n}{2}$ possible edges and they are all different, so there are $2^{\binom{n}{2}}$ different graphs, i.e. $([4], S)$ where S is any subset of the edge set. For unlabeled case, basically it is the labeled case quotient by the action of symmetric group S_n on vertices, so we can count by Burnside's lemma, i.e. for each $\sigma \in S_n$, compute the number of edge sets S which are invariant under the action of σ, denoted as N_σ, then the total number of different graphs is $\frac{\sum_{\sigma \in S_n} N_\sigma}{n!}$. Another idea is counting by degree sequences, i.e. the sequence $(a_1, a_2, \dots, a_n)$ where $a_1 \geq a_2 \geq \cdots \geq a_n$ and $a_i \in \mathbb{Z}_{\geq 0}$. We know each degree sequence corresponds to different graph, so for $k = 0, 1, \dots, \binom{n}{2}$, let N_k be the number of degree sequences with k edges, in other words the number of non-negative integer solutions of the equation $a_1 + a_2 + \cdots + a_n = 2k$ with all $a_i \leq n-1$, then the total number of different graphs is $\sum_{k=0}^{\binom{n}{2}} N_k$. (I am not sure which one is better because I don't have an idea about how hard it is to find N_k.) ✓ It's hard!

Solution 2. (i) We can have a bijection from subsets of $S_1 \cup S_2 \cup \cdots \cup S_m$ to $[a_1+1] \times \cdots \times [a_m+1]$ such that each $n_i \in [a_i+1]$ represents the action to the set S_i, for $n_i = 1, 2, \dots, a_i$ that means picking the n_i's element from S_i, and $n_i = a_i + 1$ means picking nothing from S_i. So the number of subsets of $S_1 \cup S_2 \cup \cdots \cup S_m$ is equal to $|[a_1+1] \times \cdots \times [a_m+1]| = (a_1+1)\cdots(a_m+1)$. ✓

(ii) We know any divisor d of n has the form $d = p_1^{r_1} \cdots p_m^{r_m}$ where $0 \leq r_i \leq a_i$ for $i = 1, \dots, m$. Let $S_i = \{p_i^1, \cdots, p_i^{a_i}\}$, then all S_i's are disjoint since $(p_i, p_j) = 1$ for any $i \neq j$. Each divisor d can be obtained by picking a subset of $S_1 \cup S_2 \cup \cdots \cup S_m$ containing at most one element from each set and then multiply together, apply part (a), the number of divisors of n equals $(a_1+1)\cdots(a_m+1)$. For n being a perfect square it is equivalent to say all the prime powers in its decomposition is even, and all a_i's are even if and only if all $a_i + 1$'s are odd if and only if $(a_1+1)\cdots(a_m+1)$ is odd. ✓ ✓

Solution 3. Let $S_k = \{r \in [n] : (r,n) = k\}$, for $k|n$. By our definition, S_k's are disjoint subsets of $[n]$, and $[n] = \bigcup_{k|n} S_k$. Now for any $r \in S_k$, $(r,n) = k$, or in other words $(r/k, n/k) = 1$, so each element r in S_k is one-to-one corresponded to a number r/k which is relatively prime and less or equal to n/k, then by definition of φ, $|S_k| = \varphi(n/k)$. Now $|[n]| = |\bigcup_{k|n} S_k| = \sum_{k|n} |S_k| = \sum_{k|n} \varphi(n/k)$, rewriting the formula by setting $d = n/k$, we have $n = |[n]| = \sum_{d|n} \varphi(d)$.

Solution 4. (i) Write such a k-subset as an increasing sequence, say $(a_1, a_2, \ldots, a_k)$, then we have $a_{i+1} - a_i \geq 2$ since there is no consecutive pair. We define a bijection from the set of such k-sequences of $[n]$ to the set of all strictly increasing k-sequences of $[n-k+1]$ by $(a_1, a_2, \ldots, a_k) \mapsto (b_1, b_2, \ldots, b_k)$ where $b_i = a_i - (i-1)$, $i = 1, \ldots, k$. This map is well-defined since $b_{i+1} - b_i = (a_{i+1} - i) - (a_i - (i-1)) = a_{i+1} - a_i - 1 \geq 1$ so we do get a strictly increasing sequence, and the inverse map is given by $(b_1, b_2, \ldots, b_k) \mapsto (a_1, a_2, \ldots, a_k)$ where $a_i = b_i + i - 1$. Since we have this bijection, $f(n,k)$ is equal to the number of all strictly increasing k-sequences of $[n-k+1]$ which is actually k-subsets of $[n-k+1]$, therefore we have $f(n,k) = \binom{n-k+1}{k}$.

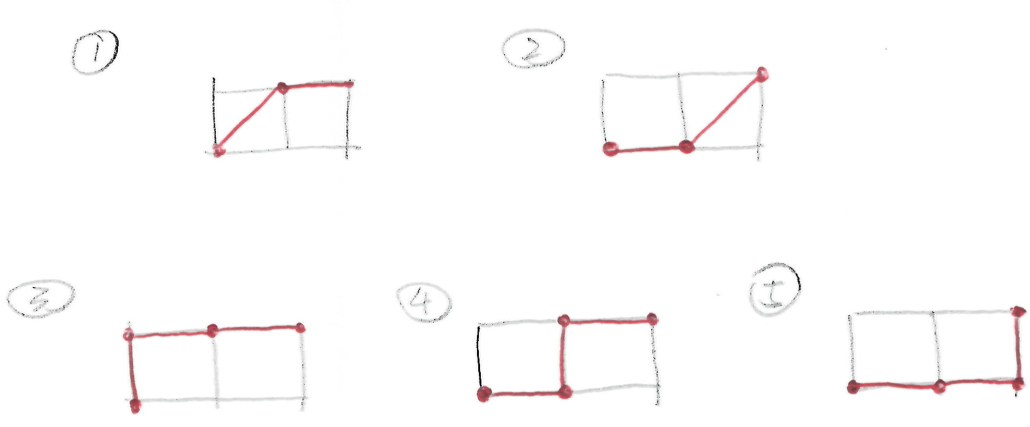
(ii) Let $T(n) = \sum_{k=0}^{n} f(n,k)$, then $T(1) = f(1,0) + f(1,1) = 1+1 = 2$, $T(2) = f(2,0) + f(2,1) + f(2,2) = 1+2+0 = 3$, now for $n > 2$, we have $T(n) = \sum_{k=0}^{n} f(n,k) = \sum_{k=0}^{n} \binom{n-k+1}{k} = \sum_{k=0}^{n} (\binom{n-k}{k} + \binom{n-k}{k-1}) = \sum_{k=0}^{n-1} \binom{n-k}{k} + \sum_{k=1}^{n-1} \binom{n-k}{k-1}$ (by Pascal's recurrence and the facts that $\binom{0}{n} = 0$ and $\binom{n}{-1} = \binom{0}{n-1} = 0$), rewriting the RHS by setting $s = k-1$, we have $T(n) = \sum_{k=0}^{n-1} \binom{(n-1)-k+1}{k} + \sum_{s=0}^{n-2} \binom{(n-2)-s+1}{s} = \sum_{k=0}^{n-1} f(n-1,k) + \sum_{s=0}^{n-2} f(n-2,s) = T(n-1) + T(n-2)$. So we have $T(1) = F_3$, $T_2 = F_4$, and $\{T(n)\}$ satisfies the same recurrence relation as $\{F_n\}$, then we must have $T(n) = F_{n+2}$ for all n.

Solution 5. If $n = 0$, then the sum has only one term $\binom{0}{0} = 1 = F_1$; when $n > 1$, apply the results in Problem 4, we have the sum of right-left diagonal is $\sum_{k=0}^{n} \binom{n-k}{k} = \sum_{k=0}^{n-1} \binom{(n-1)-k+1}{k} = \sum_{k=0}^{n-1} f(n-1,k) = T(n-1) = F_{n+1}$, the first equality is because $\binom{0}{n} = 0$ for $n > 1$.

Solution 6. (i) If $m = 0$, then we only use step $(0,1)$ therefore the last step must be $(0,1)$, so $L(0,n) = L(0,n-1)$; if $n = 0$, we only use step $(1,0)$ therefore the last step must be $(1,0)$, so $L(m,0) = L(m-1,0)$; if $m,n \geq 1$, consider the last step we use, if it is $(1,0)$, then we will arrive point $(m-1,n)$ before last step, and if it is $(0,1)$ we arrive point $(m,n-1)$, the number of paths to (m, n) is the sum of numbers of paths under this two situations, so $L(m, n) = L(m-1, n) + L(m, n-1)$. For a path from $(0, 0)$ to (m, n) using steps $(1, 0)$ and $(0, 1)$, there will be $m + n$ steps in total, actually m steps of $(1, 0)$ and n steps of $(0, 1)$, so we can have a bijection from the set of paths to all n-subsets of $[m+n]$ by taking a_i into the n-subset if the a_i's step in the path is $(0, 1)$, and the inverse map is given by mapping an n-subset S of $[n+m]$ to a path where the a_i's step is $(0, 1)$ if $a_i \in S$ and all other steps $(1, 0)$. So by the bijection, the number of all paths from $(0, 0)$ to (m, n) using steps $(1, 0)$ and $(0, 1)$ is equal $\binom{m+n}{n}$.

(ii) We prove by understanding the meaning of both sides of this equality using part (i). For RHS, $\binom{s+n+1}{s+m+1} = L(n-m, s+m+1)$, which is the number of paths from $(0, 0)$ to $(n-m, s+m+1)$. Here since $s, m, n \geq 0$, consider this point in xy coordinate, to go from $(0, 0)$ to $(n-m, s+m+1)$, we have to cross the line $y = s$. Suppose the *last* intersection point of our path and line $y = s$ is (k, s), then we can cut our path into two parts: From $(0, 0)$ to (k, s), and from (k, s) to $(n-m, s+m+1)$. (notice here I said last intersection point, because we may take steps $(1, 0)$ on the line $y = s$ so there could be more than 1 intersection points, however, if we choose (k, s) to be the last intersection point, we get unique divisions) For (k, s) being the last intersection point, it means the next step starting from (k, s) must be $(0, 1)$ and we can always take this step $(0, 1)$ because $m + s + 1$ is strictly greater than s, so by a path from (k, s) to $(n-m, s+m+1)$ actually we mean a path from $(k, s+1)$ to $(n-m, s+m+1)$ since the first step is always fixed and all other steps are free. Now counting the number of ways for each parts, the total number of paths from $(0, 0)$ to $(n-m, s+m+1)$ leaving the line $y = s$ at (k, s) is equal to the number of paths from $(0, 0)$ to (k, s) times the number of paths from $(k, s+1)$ to $(n-m, s+m+1)$, i.e. $\binom{s+k}{k}\binom{n-k}{m}$. Now summing over all possible points (k, s), we get the number of all paths from $(0, 0)$ to $(n-m, s+m+1)$, i.e. $\sum_{k=0}^{n} \binom{s+k}{k}\binom{n-k}{m} = L(n-m, s+m+1) = \binom{s+n+1}{s+m+1}$.

Solution 7. (i) There are two cases: we use diagonal step or we don't use diagonal step. In the first case, there will only be two steps to reach $(2, 1)$, one $(1, 1)$ and one $(1, 0)$, we only need to decide which step is the diagonal step, so there are two paths; in the second case, it is just the same as the path problem in Problem 6, so there are $\binom{3}{1} = 3$ paths. They look like:



(ii) With a fixed number, say l, of diagonal steps, the number of Delannoy paths is $\binom{m+n-l}{n-l,m-l,l}$, since there are $m+n-l$ steps in total (besides l diagonal steps, there are also $m-l$ steps of $(1,0)$ and $n-l$ steps of $(0,1)$), and we need to choose which step to take $(1,0)$, which step to take $(0,1)$ and which step to take $(1,1)$. Summing over all possible numbers of digaonal steps, we have $D_{m,n} = \sum_{l=0}^{min(m,n)} \binom{m+n-l}{n-l,m-l,l} = \sum_{l=0}^{min(m,n)} \binom{m+n-l}{n-l}\binom{m}{m-l} = \sum_{l=0}^{min(m,n)} \binom{m+n-l}{m}\binom{m}{m-l}$, setting $k = m-l$, we have $D_{m,n} = \sum_{k=m-min(m,n)}^{m} \binom{n+k}{m}\binom{m}{k}$.

Remark: I discussed most of the problems with Aziz and Jonghoo.