Math 6501 - Enumerative Combinatorics I – Homework 2

Due at 3:00pm on Monday September 16th, 2019

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams, but individual solutions **must** be submitted for grading and credit. If you work in teams, please indicate the name of your collaborator(s) for each problem.

In Problems 2 through 7 below, q denotes a fixed *formal parameter.*

Problem 1. (**A combinatorial interpretation of q-factorial numbers**)

(i) Show that the number of sequences $\emptyset = S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_n = \{1, \ldots, n\}$ of subsets of $[n]$ is $n!$.

(ii) Let q be a prime power, i.e., $q = p^m$, and let $\mathbf{F}_q$ be the finite field with q-elements. A *full- flag* of vectors subspaces of $\mathbf{F}_q$ is as sequences of the form:

$$\mathcal{F}_\bullet \colon \{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_n = \mathbf{F}_q^n,$$

where each V_i is a vector subspace of $\mathbf{F}_q^n$. Show that the number of such full flags is $[n]_q!$.

Problem 2. Prove the following variant of Pascal's q-recurrence:

$$\left[{n \atop k}\right]_q = q^k \left[{n-1 \atop k}\right]_q + \left[{n-1 \atop k-1}\right]_q \quad \text{for } k, n \in \mathbb{Z}_{\geq 0},$$

where $\left[{0 \atop 0}\right]_q = 1$ and $\left[{n \atop k}\right]_q = 0$ for $k > n$. Use it to show $\left[{n \atop k}\right]_q$ in $\mathbb{Z}[q]$ has degree $k(n-k)$ if $0 \leq k \leq n$.

Problem 3. Let m, n, k be non-negative integers and consider the primitive nth. root of unity $\zeta = e^{2\pi i/n} = \cos(\frac{2\pi}{n}) + i \sin(\frac{2\pi}{n})$ in $\mathbb{C}$. Show that the polynomial $f(q) = \left[{nm \atop k}\right]_q$ in $\mathbb{Z}[q]$ satisfies:

$$f(\zeta) = \begin{cases} \binom{m}{\ell} & \text{if } k = n\,\ell \text{ for some } \ell \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 4. Let $0 \leq k \leq n$ and set $f(q) := \left[{n \atop k}\right]_q \in \mathbb{Z}[q]$. Show that $f(1/q) = q^{-k(n-k)}\, f(q)$ and use it to prove $f'(1) = \frac{k(n-k)}{2} \binom{n}{k}$.

Problem 5. Proved the following q-analog of the **Binomial Theorem**:

$$\prod_{k=0}^{n-1} (1 + q^k\, x) = \sum_{k=0}^{n} q^{\binom{k}{2}} \left[{n \atop k}\right]_q x^k \qquad \text{where } n \in \mathbb{Z}_{>0} \text{ is fixed.}$$

Problem 6. Prove the q-**Vandermonde identity**:

$$\sum_{i=0}^{\ell} q^{i(i+m-\ell)} \left[{n \atop i}\right]_q \left[{m \atop \ell-i}\right]_q = \left[{n+m \atop \ell}\right]_q \qquad \text{where } m, n, \ell \in \mathbb{Z}_{\geq 0}.$$

(*Hint:* Mimic the proof of the classical Vandermonde identity discussed in class.)

Problem 7. (**Non-commuting q-Binomial and q-Multinomial Theorems**)

(i) Let x and y be non-commuting variables satisfying the commutation relation $yx = qxy$. Assume q commutes with x and y. Show that

$$(x + y)^n = \sum_{k=0}^{n} \left[{n \atop k}\right]_q x^k y^{n-k}.$$

(ii) Let $m \geq 1$. Consider m variables $x_1, \ldots, x_m$ subject to the commuting relation: $x_i x_j = q\, x_j x_i$ for all $i > j$. Assume q commutes with all x_i's. Generalize the identity of the previous item to $(x_1 + \ldots + x_m)^n$.

(iii) [**Bonus**] Let $m \geq 1$. Consider m *commuting* parameters $q_1, \ldots, q_m$ and m *non-commuting* variables $x_1, \ldots, x_m$ subject to the commuting relation: $x_i x_j = q_j x_j x_i$ for all $i > j$ and $q_k x_l = x_l q_k$ for all k, l. Generalize the identity of the previous item to $(x_1 + \ldots + x_m)^n$ in this new setting.