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Combinatorics - I
Homework - II

Problem 1:

(i) Observe that given a seq $\emptyset = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n = [n]$ we have $|S_i| = i$

Proof:

$|S_i| \geq i$:

$|S_0| = |\emptyset| = 0 \geq 0$.

Assume $|S_{i-1}| \geq i-1$

then, $|S_i| = |S_i \setminus S_{i-1}| + |S_{i-1}|$

we know that $S_{i-1} \subsetneq S_i$. So, $|S_i \setminus S_{i-1}| \geq 1$

So, $|S_i| = |S_i \setminus S_{i-1}| + |S_{i-1}| \geq 1 + (i-1) = i$.

$|S_i| \leq i$:

Take the complement of the sets in the seq in order to get

$\emptyset = S_n^c \subsetneq S_{n-1}^c \subsetneq \cdots \subsetneq S_1^c \subsetneq S_0^c = [n]$.

by the argument above $|S_{n-i}^c| \geq i$.

So $|S_{n-i}| \leq n-i \Rightarrow |S_i| \leq i$

$\Rightarrow |S_i| = i$.

So, in order to form such a seq we have

n - many choices for S_1
$(n-1)$ - many choices for S_2
$\vdots$
2 - many choices for S_{n-1}
1 choice for S_n

$\longrightarrow$ So, number of such sequences is $= n \cdot (n-1) \cdots 2 \cdot 1 = n!$

(ii) By a similar argument as in part (i) and with some basic linear algebra, we get $\dim(V_i) = i$ if

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{F}_q^n$$

To form V_1, we have $\frac{q^n-1}{q-1}$ $(= [n]_q)$-many choices. ✓

($q^n - 1$ = number of non-zero vectors. And we are dividing this number by $q-1$ = (number of non zero scalars) because v and λv span the same subspace when $v \in \mathbb{F}_q^n - \{0\}$ & $\lambda \in \mathbb{F}_q - \{0\}$. ✓)

To form V_2, we have $\frac{q^n - q}{q^2 - q}$ $(= [n-1]_q)$-many choices

($q^n - q = |\mathbb{F}_q^n \setminus V_1|$. And we are dividing this number by $q^2 - q$ because when pick $v \in \mathbb{F}_q^n \setminus V_1$, $|\text{span}(V_1 \cup \{v\})| = q^2$ and for any $\tilde{v} \in \text{span}(V_1 \cup \{v\}) \setminus V_1$, we will have $\text{span}(V_1 \cup \{v\}) = \text{span}(V_1 \cup \{\tilde{v}\})$. So, we don't want to over count the number of subspaces. For this reason, we are dividing by $q^2 - q = |\text{span}(V_1 \cup \{v\}) \setminus V_1|$. ✓)

Similarly, to form V_i, we have $\frac{q^n - q^{i-1}}{q^i - q^{i-1}} = [n-(i-1)]$-many choices. So, the total number of full-flags of $\mathbb{F}_q^n$ is $= [n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q = [n]_q!$ $\square$. ✓

Problem 2: Prove that $\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$.

Proof 1:

$$q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = q^k \cdot \frac{[n-1]_q!}{[k]_q! \cdot [n-k-1]_q!} + \frac{[n-1]_q!}{[k-1]_q! \cdot [n-k]_q!}$$

$$= \frac{q^k [n-1]_q! \cdot [n-k] + [n-1]_q! [k]_q}{[k]_q! \cdot [n-k]_q!}$$

$$= \frac{[n-1]_q! \left(q^k [n-k] + [k]_q \right)}{[k]_q! [n-k]_q!} = \frac{[n-1]_q! \left(q^k (1+q+\dots+q^{n-k-1}) + (1+q+\dots+q^{k-1}) \right)}{[k]_q! [n-k]_q!}$$

$$= \frac{[n-1]_q! \cdot (1+q+\dots+q^{n-1})}{[k]_q! \cdot [n-k]_q!} = \frac{[n-1]_q! \cdot [n]_q}{[k]_q! \cdot [n-k]_q!} = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

$\square$.

Proof 2:

In the class we proved that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

So, $$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_q + q^{n-(n-k)} \begin{bmatrix} n-1 \\ (n-k)-1 \end{bmatrix}$$

$$= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}$$

$\square$.

$\begin{bmatrix} n \\ k \end{bmatrix}_q$ has degree $k(n-k)$ when $0 \leq k \leq n$:

First observe that when $n=0$ or $n=k$ or $k=0$, we have $\begin{bmatrix} n \\ k \end{bmatrix} = 1$

So, its degree is $k.(n-k) = 0$ (in both cases). ✓

Now we can assume $0 \leq k < n$ and we can do induction on n.

The base case: already done above. ✓

Induction hypothesis: Assume $\begin{bmatrix} m \\ k \end{bmatrix}$ has degree $k(m-k)$ if $0 \leq k \leq m < n$.

Now $\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$

By induction hypothesis, $\begin{bmatrix} n-1 \\ k \end{bmatrix}_q = a_0 + a_1 q + \cdots + a_{k(n-k-1)} q^{k(n-k-1)}$

with $a_{k(n-k-1)} \neq 0$

& $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = b_0 + b_1 q + \cdots + b_{(k-1)(n-k)} q^{(k-1)(n-k)}$ w/ $b_{(k-1)(n-k)} \neq 0$. ✓

So, $\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k (a_0 + a_1 q + \cdots a_{k(n-k-1)} q^{k(n-k-1)}) + (b_0 + b_1 q + \cdots + b\, q^{(k-1)(n-k)})$

So, the leading term of $\begin{bmatrix} n \\ k \end{bmatrix}_q = a_{k(n-k-1)} \cdot q^{k+k(n-k-1)}$ ($k+k(n-k-1) > (k-1)(n-k)$ for $k<n$ ✓)

$= a_{k(n-k-1)} \cdot q^{k(n-k)}$

So, degree of $\begin{bmatrix} n \\ k \end{bmatrix}_q = k(n-k)$ ✓

□

Problem 3: Let $n, m, k \in \mathbb{Z}_{\geq 0}$, $\zeta = e^{\frac{2\pi i}{n}} \in \mathbb{C}$. $f(q) = \left[{nm \atop k} \right]_q$

Show that

$$f(\zeta) = \begin{cases} \binom{m}{\ell} & \text{if } k = n.\ell \text{ for some } \ell \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

Proof: (This proof is inspired from page 190, Enumerative Combinatorics, Volume 1, 2^{nd} edition, by Richard Stanley).

By problem 5, we have $\prod_{i=0}^{nm-1} (1 + \zeta^i(-x)) = \sum_{i=0}^{nm} (\zeta)^{\binom{i}{2}} \left[{nm \atop i} \right]_\zeta (-x)^i$

Observe that $\prod_{i=0}^{nm-1} (1+\zeta^i(-x)) = \prod_{i=0}^{nm-1} (1 - \zeta^i x)$ $\left[\zeta^{n+\ell} = \zeta^\ell \right]$

$$= \underbrace{(1-x)(1-\zeta x) \cdots (1-\zeta^{n-1} x)}. \underbrace{(1-x)(1-\zeta x) \cdots (1-\zeta^{n-1} x)}. \cdots . \underbrace{(1-x)(1-\zeta x) \cdots (1-\zeta^{n-1} x)}$$

$$= \underbrace{(1-x^n) \cdot (1-x^n) \cdots (1-x^n)}_{m\text{-many}}$$

$$= (1-x^n)^m.$$

So, we have $(1-x^n)^m = \sum_{i=0}^{nm} (\zeta)^{\binom{i}{2}} \left[{nm \atop i} \right]_\zeta (-1)^i x^i = \sum_{i=0}^{nm} (-1)^i (\zeta)^{\binom{i}{2}} . f(\zeta) x^i$

$$\Rightarrow \sum_{i=0}^{m} \binom{m}{i} (-1)^i (x^n)^i = (1-x^n)^m = \sum_{i=0}^{nm} (-1)^i (\zeta)^{\binom{i}{2}} f(\zeta) x^i$$

Let's compare the coefficents of x^i's in both sides of the equation.

On the left hand side the coefficient of x^k is nonzero if and only if $k = n.l$ for some $l \in \{0, 1, \dots, m\}$. ✓

On the right hand side, the coeff. of x^k is $(-1)^k . (\zeta)^{\binom{k}{2}} . f(\zeta)$.

So, $f(\zeta) = 0$ if and only if $k \neq n.l$ for some $l \in \{0, 1, \dots, m\}$. ✓

If $k = n.l$ for some $l \in \{0, 1, \dots, m\}$. Then, on the left side the coeff of x^{nl} is $(-1)^l . \binom{m}{l}$

So, $(-1)^l \binom{m}{l} = (-1)^{nl} (\zeta)^{\binom{nl}{2}} . f(\zeta)$.

$$\Rightarrow f(\zeta) = \frac{(-1)^{l-nl}}{(\zeta)^{\binom{nl}{2}}} . \binom{m}{l}$$

n = even, l = even
n = odd, l = even
n = odd, l = odd
$\Rightarrow l - nl$ = even & $n \mid \binom{nl}{2} \Rightarrow f(\zeta) = \binom{m}{l}$ ✓

n = even, l = odd $\Rightarrow (-1)^{l-nl} = -1$, $\binom{nl}{2} = \frac{n}{2} . (2t-1)$ for some $t \in \mathbb{Z}_{>0}$

$$\Rightarrow (\zeta)^{\binom{nl}{2}} = \left((\zeta)^{\frac{n}{2}}\right)^{2t-1} = (-1)^{2t-1} = -1$$

$$\Rightarrow f(\zeta) = \frac{-1}{-1} \binom{m}{l} = \binom{m}{l}$$ ✓

So, $f(\zeta) = \begin{cases} \binom{m}{l} & \text{if } k = n.l \text{ for some } l \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$ ✓ ✓

Lovely!

$\square$.

Problem 4:

$f(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$, $0 \le k \le n$. Show that $f\left(\frac{1}{q}\right) = q^{-k(n-k)} f(q)$.

Induction on degree on f:

Base case: $\deg(f) = 0$.

By problem 2, $\deg(f) = 0$ tells us that either $k = 0$ or $n = k$.

In each case, we get $f(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q = 1$.

So, $f\left(\frac{1}{q}\right) = 1 = q^0 \cdot 1 = q^0 \cdot f(q) = q^{-k(n-k)} \cdot f(q)$.

Inductive hypothesis: Assume that $g(q) = \begin{bmatrix} m \\ l \end{bmatrix}_q$ $0 \le l \le m$

satisfies $g\left(\frac{1}{q}\right) = q^{-l(m-l)} \cdot g(q)$ when $\deg(g) < k(n-k) = \deg(f)$.

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \underset{\text{proved in class}}{=} \begin{bmatrix} n \\ k \end{bmatrix}_q = f(q) \underset{\text{problem 2}}{=} q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

For simplicity, let $\begin{bmatrix} n-1 \\ k \end{bmatrix}_q = h(q)$ & $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = t(q)$.

Then $q^k \cdot f(q) = q^k \cdot h(q) + q^n t(q)$

$= \left(f(q) - t(q) \right) + q^n t(q)$

$\Rightarrow (q^k - 1) f(q) = (q^n - 1) \cdot t(q)$

$\Rightarrow f(q) = \frac{q^n - 1}{q^k - 1} \cdot t(q)$.

$t(q) = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$ so, $\deg(t) = (k-1)(n-k) < k(n-k) = \deg(f)$

So, by induction hypothesis $t\left(\frac{1}{q}\right) = q^{-(k-1)(n-k)} \cdot t(q)$ ✓

So, $f\left(\frac{1}{q}\right) = \dfrac{\left(\frac{1}{q}\right)^n - 1}{\left(\frac{1}{q}\right)^k - 1} \cdot t\left(\frac{1}{q}\right) = q^{k-n} \cdot \dfrac{q^n - 1}{q^k - 1} \cdot q^{-(k-1)(n-k)} \cdot t(q)$ ✓

$$= q^{-k\cdot(n-k)} \cdot \frac{q^n-1}{q^k-1} \cdot t(q)$$

$$= q^{-k(n-k)} \cdot f(q)$$ ✓

$\square$.

Applying $\frac{d}{dq}$ to both sides of the eqn $f\left(\frac{1}{q}\right) = q^{-k(n-k)} f(q)$

we get, $f'\left(\frac{1}{q}\right) \cdot \left(-\frac{1}{q^2}\right) = (-k(n-k)) q^{-k(n-k)-1} \cdot f(q) + f'(q) \cdot q^{-k(n-k)}$ ✓

By evaluating at $q=1$, we get ✓

$-f'(1) = (-k(n-k)) \cdot f(1) + f'(1)$

$\Rightarrow 2f'(1) = k(n-k) f(1) = k(n-k) \cdot \begin{bmatrix} n \\ k \end{bmatrix}_1 = k(n-k)\binom{n}{k}$

So, $f'(1) = \dfrac{k(n-k)}{2}\binom{n}{k}$ ✓

$\square$

Problem 5: Prove the q-analog of the Binomial Theorem

$$\prod_{k=0}^{n-1}(1+q^k x) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$$

Proof: Induction on n:

For $n=1$ on both sides we have $1+x$, so it is true.

Assume the equality holds for $n-2$.

$\left(\text{That is } \prod_{k=0}^{n-2}(1+q^k x) = \sum_{k=0}^{n-1} q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix} x^k\right)$

$$\text{Then } \prod_{k=0}^{n-1}(1+q^k x) = \left(\prod_{k=0}^{n-2}(1+q^k x)\right)(1+q^{n-1}x)$$

$$= \left(\sum_{k=0}^{n-1} q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q x^k\right)(1+q^{n-1}x)$$

$$\binom{k}{2} = \binom{k-1}{2} + \binom{k-1}{1}$$

$$\Rightarrow$$

$$\binom{k-1}{2} = \binom{k}{2} - (k-1)$$

$$= \sum_{k=0}^{n}\left(q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-1} \cdot q^{\binom{k-1}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q\right) x^k$$

$$= \sum_{k=0}^{n}\left(q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{\binom{k}{2}} \cdot q^{(n-1)-(k-1)} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q\right) x^k$$

$$= \sum_{k=0}^{n} q^{\binom{k}{2}}\left(\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q\right) x^k = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \quad \square$$

Problem 6: Prove the q-Vondermonde Identity

$$\sum_{k=0}^{n+m} q^{\binom{k}{2}} \begin{bmatrix} n+m \\ k \end{bmatrix}_q x^k = \prod_{k=0}^{n+m-1} (1+q^k x) = \prod_{k=0}^{m-1} (1+q^k x) \cdot \prod_{k=m}^{n+m-1} (1+q^k x)$$

$$= \left(\sum_{k=0}^{m} q^{\binom{k}{2}} \begin{bmatrix} m \\ k \end{bmatrix}_q x^k \right) \cdot \prod_{k=0}^{n-1} (1+q^{k+m} x) = \left(\sum_{k=0}^{m} q^{\binom{k}{2}} \begin{bmatrix} m \\ k \end{bmatrix}_q x^k \right) \cdot \prod_{k=0}^{n-1} (1+q^k (q^m x))$$

$$= \left(\sum_{k=0}^{m} q^{\binom{k}{2}} \begin{bmatrix} m \\ k \end{bmatrix}_q x^k \right) \cdot \left(\sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (q^m x)^k \right) \checkmark$$

By comparing the coefficients of x^l in each sides of the equation, we get

$$q^{\binom{l}{2}} \begin{bmatrix} n+m \\ l \end{bmatrix}_q = \sum_{i=0}^{l} q^{\binom{l-i}{2}} \begin{bmatrix} m \\ l-i \end{bmatrix}_q \cdot q^{\binom{i}{2}} \cdot \begin{bmatrix} n \\ i \end{bmatrix} (q^m)^i \checkmark$$

Dividing both sides by $q^{\binom{l}{2}}$ gives us:

$$\begin{bmatrix} n+m \\ l \end{bmatrix}_q = \sum_{i=0}^{l} q^{\left[\binom{l-i}{2} + \binom{i}{2} + mi - \binom{l}{2}\right]} \cdot \begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} m \\ l-i \end{bmatrix}_q$$

$$= \sum_{i=0}^{l} q^{i(i+m-l)} \cdot \begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} m \\ l-i \end{bmatrix}_q \checkmark \quad \square$$

$$\binom{l-i}{2} + \binom{i}{2} + mi - \binom{l}{2} = \frac{(l-i)(l-i-1)}{2} + \frac{i(i-1)}{2} - \frac{l(l-1)}{2} + mi$$

$$= \frac{\cancel{l^2} - \cancel{li} - \cancel{l} - il + i^2 + \cancel{i} + i^2 - \cancel{i} - \cancel{l^2} + \cancel{l}}{2} + mi = i^2 - il + mi$$

$$= i(i+m-l) \checkmark$$

Problem 7:

(i) Given $yx = qxy$, Prove that $(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}$

Proof: Induction on n

for $n=0$, both sides are 1. So, the equality holds.
Assume the equality holds for $n-1$. Then

$$(x+y)^n = (x+y)^{n-1}(x+y) = \left(\sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q x^k y^{n-1-k}\right)(x+y)$$ ✓

$$= \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q x^k y^{n-1-k} \cdot x + \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q x^k y^{n-k}$$ ✓

$$= \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{n-1-k} x^{k+1} y^{n-1-k} + \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q x^k y^{n-k}$$

$$= \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q q^{n-k} x^k y^{n-k} + \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q x^k y^{n-k}$$ ✓

$$= \sum_{k=0}^{n} \left(\left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q q^{n-k} + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \right) x^k y^{n-k} \right) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}$$ ✓ $\square$

(ii) Given $x_i \cdot x_j = q\, x_j x_i$ for $i > j$, we have

$$(x_1 + \cdots + x_m)^n = \sum_{\substack{a_i \geq 0 \\ a_1 + \cdots + a_m = n}} \begin{bmatrix} n \\ a_1, \ldots, a_m \end{bmatrix} x_1^{a_1} \cdots x_m^{a_m}.$$ ✓

Proof of this equality comes from the following claim

Claim: $$\begin{bmatrix} n \\ a_1, \ldots, a_m \end{bmatrix} = q^{n-a_1} \begin{bmatrix} n-1 \\ a_1-1, a_2, \ldots, a_m \end{bmatrix} + q^{n-a_1-a_2} \cdot \begin{bmatrix} n-1 \\ a_1, a_2-1, \ldots, a_m \end{bmatrix} + \cdots$$

$$+ \cdots q^{n-a_1-a_2-a_{m-1}} \begin{bmatrix} n-1 \\ a_1, \ldots, a_{m-1}-1, a_m \end{bmatrix} + \begin{bmatrix} n-1 \\ a_1, \ldots, a_m - 1 \end{bmatrix}$$ ✓

Proof of this claim is the same as when $m=2$

Start from RHS, equalize the denominators, and get the result.

Proof of the generalized formula is again by induction

and it is almost identical to the proof in part (i). ✓