Math 6501 - Enumerative Combinatorics I – Homework 3

Due at 3:00pm on Monday September 30th, 2019

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams, but individual solutions **must** be submitted for grading and credit. If you work in teams, please indicate the name of your collaborator(s) for each problem.

Problem 1. Fix two integers m, n with $n > m$. Prove that $\sum_{k=1}^{n} (-1)^k \binom{n}{k} k^m = 0$.
(*Fun fact:* If $n > m - \varphi(m)$, where φ is Euler's φ-function, one can show that $m | \sum_{k=1}^{n} (-1)^k \binom{n}{k} k^m$ in $\mathbb{Z}$.)

Problem 2. How many integer solutions $x_1 + x_2 + x_3 + x_4 = 30$ exist with the restriction $-10 \leq x_i \leq 20$?

Problem 3. Let $C(n, k, s)$ be the number of k-subsets of $[n] = \{1, 2, \ldots, n\}$ that contain no run of s consecutive integers. Show that

$$C(n, k, s) = \sum_{i=0}^{\lfloor k/s \rfloor} (-1)^i \binom{n-k+1}{i} \binom{n - i\,s}{n-k}.$$

(*Hint:* Recall that the number of non-negative integer solutions to $x_1 + \ldots + x_m = \ell$ with $x_i < s$ for all i equals $\sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{m+\ell-js-1}{m-1}$.)

Problem 4. (i) Show that the number of ways in which n male-female couples can be seated on a long dinner table so that no couple sits next to each other is $\sum_{k=0}^{n} (-2)^k \binom{n}{k} (2n-k)!$.

(*Hint:* Take X to be the set of all seatings, so $|X| = (2n)!$, and define e_i as the property that the i-th couple sits side by side.)

(ii) Show that also imposing that women and men must alternate seating yields $2\,n! \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} (n-k)!$ many seating arrangements.

Problem 5. (i) Let A be a finite set and consider n subsets $A_1, \ldots, A_n$ of A. Given a subset T of $[n]$, we define $A_T = \bigcap_{i \in T} A_i$ and set $S_k = \sum_{|T|=k} |A_T|$ for each $k = 0, \ldots, n$. Show that $\sum_{i=k}^{n} (-1)^{i-k} S_i = S_k - S_{k+1} + \ldots + (-1)^{n-k} S_n \geq 0$ for all $k = 0, \ldots, n$.

(ii) **(Bonus)** Prove the following statement: a vector $(S_0, \ldots, S_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ can be realized from n subsets $A_1, \ldots, A_n$ of a finite set A as in item (i) if and only if $\sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} S_i \geq 0$ for all $k = 0, \ldots, n$.

Problem 6. Show, by means of an example, that the involution used in the proof of the Linström-Gessel-Viennot Lemma (from Lecture 10) does not work if we just choose i_0 minimal, then the minimal j_0 such that the paths P_{i_0}, P_{j_0} intersect, and then the first common point X on P_{i_0}. What could go wrong?

Problem 7. (Properties of determinants via Lindström-Gessel-Viennot's Lemma)

(i) Show that any $n \times m$ matrix M corresponds to the path matrix between two sets $A = \{A_1, \ldots, A_n\}$ and $B = \{B_1, \ldots, B_m\}$ corresponding to the vertices of a bipartite graph G with $\mathrm{wt}(A_i \to B_j) = m_{ij}$.

(ii) Use LGV to show that $\det M^T = \det M$.

(iii) Given two $n \times n$ matrices M and M', use LGV to show that $\det(M\,M') = \det(M) \det(M')$, (*Hint:* Consider three sets $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n)$, $C = (C_1, \ldots, C_n)$ and identify M and M' with the path matrices of bipartite graphs (A, B) and (B, C) as in item (i).)

(iv) **(Cauchy-Binnet)** Let M be an $n \times p$-matrix and M' be an $p \times n$-matrix with $n \leq p$. Then

$$\det(MM') = \sum_{|R|=n} \det M_R \, \det M'_R,$$

where M_R is the $n \times n$ submatrix of M with columns in R and M'_R is the corresponding submatrix of M' with rows in R. (*Hint:* Use the same strategy as in item (ii), but now set the length of B to be p.)