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MATH 6501: HW 3

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1. Fix two integers m, n with $n > m$. Prove that $\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^m = 0$.

   Proof. Suppose we have m distinct balls and n distinct boxes. Let X be the set of all possibilities of distributing balls into boxes. Let e_i be a property of X such that $x \in X$ uses i-th box. This is clearly homogeneous property and $N_{\geq k} = k^m$. As $n > m$, there is no way of distributing m balls into n boxes. Therefore,

   Not clear

   e_i: property$_{i\text{th}}$:
   Better: doesn't use i box
   $N_{\geq k} = (n-k)^m$

   $$0 = N_{=\emptyset} = \sum_{k=1}^{n}(-1)^k\binom{n}{k}k^m.$$

   □

2. How many integer solutions $x_1 + x_2 + x_3 + x_4 = 30$ exist with $-10 \leq x_i \leq 20$?

   Proof. For any integer solutions $x_1 + x_2 + x_3 + x_4 = 30$ with the restriction $-10 \leq x_i \leq 20$, consider $y_i = x_i + 10$. Then, $y_1 + y_2 + y_3 + y_4 = 70$ and $0 \leq y_i \leq 30$. Conversely, any integer solutions $y_1 + y_2 + y_3 + y_4 = 70$ with the restriction $0 \leq y_i \leq 30$ correponds to integer solutions $x_1 + x_2 + x_3 + x_4 = 30$ with $-10 \leq x_i \leq 20$. Thus, it suffices to count the number of weak 4-compositions of 30 such that each $y_i \leq 30$, or equivantly each $y_i < 31$. But, we know what this number should be;

   $$\sum_{k=0}^{4}(-1)^k\binom{4}{k}\binom{73-31k}{3} = \binom{73}{3} - 4\binom{42}{3} + 6\binom{11}{3} = 17266.$$

   □

3. Let $C(n, k, s)$ be the number of k-subsets of $[n]$ that contain no run of s consecutive integers. Show that

   $$C(n, k, s) = \sum_{i=0}^{\lfloor k/s \rfloor}(-1)^i\binom{n-k+1}{i}\binom{n-is}{n-k}.$$

   (*Hint: Recall that the number of non-negative integer solutions to* $x_1 + \cdots + x_m = l$ *with* $x_i < s$ *for all* i *equals* $\sum_{j=0}^{m}(-1)^j\binom{m}{j}\binom{m+l-js-1}{m-1}$.)

Ex. $n=5$, $k=2$, $s=2$, $A=\{1,5\}$, $A^c=\{2,3,4\}$

Proof. For each k-subset A of $[n]$ with no run of s consecutive integers, consider its complement A^c. Then, $|A^c| = n-k$ and there are $n-k+1$ places we used to put bars. Call such run of integers a bar. (See Figure). For each bar, we count the number of integers appearing in the bar. Then, this number is always strictly less than s, and the sum of all such numbers equals k. Thus, we are counting the number of weak $n-k+1$-compositions of k with the restriction $x_i < s$. By hint, we know this number is

$$\sum_{j=0}^{m}(-1)^j\binom{n-k+1}{j}\binom{n-k+1+k-js-1}{n-k+1-1} = \sum_{j=0}^{m}(-1)^k\binom{n-k+1}{j}\binom{n-js}{n-k}.$$

Since $\binom{n-js}{n-k} = 0$ if $n-js < n-k$, we can truncate this sum by only considering when $k/s \geq j$. But as j should be an integer, we can sharpen the bound to be $\lfloor k/s \rfloor$. □

4. (a) Show that the number of ways in which n male-female couples can be seated on a long dinner table so that no couple sits next to each other is $\sum_{k=0}^{n}(-2)^k\binom{n}{k}(2n-k)!$.

   (b) Show that also imposing that women and men must alternate seating yields $2n!\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}(n-k)!$ many seating arrangements.

Proof. (a) Let X to be the set of all possible seatings. So $|X| = (2n)!$. Define e_i to be a property for X such that i-th couple sits next to each other. Then, this property is homogeneous, and we are looking for the number $N_{=\emptyset}$. Now,

$$N_{=\emptyset} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}N_{\geq k}.$$

To count $N_{\geq k}$, for any seating contributing to $N_{\geq k}$, there are at least k couples sitting next to each other. Starting from one side of the table, (say left to right), define the sequence whose i-entry is the position of the i-couple sitting next to each other where the position of a couple is defined to be the smaller position between two. Then, we have $\{p_1, \ldots, p_k\} \subseteq [2n-1]$ with no two consecutive integers.
(Problem 4) From HW 1, we know this number equals $\binom{2n-1-k+1}{k} = \binom{2n-k}{k}$. Now, this k happy seats can be assigned to happy couple in $k!$ many ways and these happy couple can sit in two ways (ladies first or gentlemen first), and finally the remaining sad people can sit arbitarily in $2n-2k$ seats, we have $N_{\geq k} = \binom{2n-k}{k}k!2^k(2n-2k)! = (2n-k)!2^k$.
Thus,

$$N_{=\emptyset} = \sum_{k=0}^{n}(-2)^k\binom{n}{k}(2n-k)!$$

(b) With the same notation above, we count $N_{\geq k}$ again so that we count only seating assignment satisfying the additional requirement. After choosing the seat assignments for those who sit side by side ($\binom{2n-k}{k}$ choices), we ask each man of the sad couples (not sitting side by side) to choose their seat only from the seats with odd positions ($(n-k)!$), then the happy couple choose their seat from the assigned seats ($k!$) Finally, the each woman from the sad couple choose their seat from the remaining seats $(n-k)!$. Note that we may repeat this procedure with requiring the sad men to choose even position seats. Hence $N_{\geq k} = 2((n-k)!)^2\binom{2n-k}{k}k!$. Now the result follows by the PIE.

$\square$

5. (a) Let A be a finite set and consider n subsets $A_1, \ldots, A_n$ of A. Given a subset T of $[n]$, we define $A_T = \cap_{i \in T} A_i$ and set $S_k = \sum_{|T|=k} |A_T|$ for each $k = 0, \ldots, n$. show that $\sum_{i=k}^{n}(-1)^{i-k}S_i = S_k - S_{k+1} + \cdots + (-1)^{n-k}S_n \geq 0$ for all $k = 0, \ldots, n$.

   (b) Prove the following statement: a vector $(S_0, \ldots, S_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ can be realized from n subsets $A_1, \ldots, A_n$ of a finite set A as in item (i) if and only if $\sum_{i=1}^{k}(-1)^{i-k}\binom{i}{k}S_i \geq 0$ for all $k = 0, \ldots, n$.

Proof. (*I had to look at the solutions given in Stanley's book. I tried to fill all of the missing details.*)

Consider each A_i as the set of elements in A satisfying a property e_i. (We don't need to define what e_i is.) Then, $|A_T| = N_{\supseteq T}$, and we know $N_{\supseteq T} = \sum_{B, T \subseteq B} N_{=B}$. Fix $k \in \{0, 1, \ldots, n\}$, then we can write by def.

$$\sum_{i=k}^{n}(-1)^{i-k}S_i = \sum_{T, |T| \geq k}(-1)^{|T|-k}|A_T|$$

by expanding each S_i. As we noted above, each $|A_T| = \sum_{B, T \subseteq B} N_{=B}$. So substitute each $|A_T|$ with this expression so that we have

$$\sum_{i=k}^{n}(-1)^{i-k}S_i = \sum_{T, |T| \geq k}\sum_{B, T \subseteq B}(-1)^{|T|-k}N_{=B}.$$

We want to switch the order of the sums, since any T appearing in the summation has at least k elements, we see that each B in the second sum has at least k elements as well. So, if we reindex B's by their cardinality, we must have

$$\sum_{T, |T| \geq k}\sum_{B, T \subseteq B}(-1)^{|T|-k}N_{=B} = \sum_{B, |B| \geq k}\sum_{\substack{T, T \subseteq B \\ |T| \geq k}}(-1)^{|T|-k}N_{=B}$$

$$= \sum_{B,|B|\geq k} N_{=B} \sum_{\substack{T, T\subseteq B \\ |T|\geq k}} (-1)^{|T|-k}.$$

Finally, reindex the second sum by the cardinality of each T, that is,

$$\sum_{B,|B|\geq k} N_{=B} \sum_{\substack{T, T\subseteq B \\ |T|\geq k}} (-1)^{|T|-k} = \sum_{B,|B|\geq k} N_{=B} \sum_{i=k}^{|B|} (-1)^{i-k} \sum_{T,|T|=i} 1$$

$$= \sum_{B,|B|\geq k} N_{=B} \sum_{i=k}^{|B|} (-1)^{i-k} \binom{|B|}{k}. \checkmark$$

Note that $\sum_{i=k}^{n}(-1)^{i-k}\binom{n}{k} = \binom{n-1}{k-1}$ for any n. This can be shown by writing each $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, then the sum forms a telescoping sum with the last term $(-1)^n - k\binom{n-1}{n} = 0$. In our case, $|B| \geq k$, so $\sum_{i=k}^{|B|}(-1)^{i-k}\binom{|B|}{k} = \binom{|B|-1}{k-1} \geq 0$ always. As $N_{=B} \geq 0$ by definition, this shows that $\sum_{i=k}^{n}(-1)^{i-k} S_i \geq 0$. ✓ ✓ □

6. Show, by means of example, that the involution used in the proof of the Linstróm-Gessel-Viennot Lemma does not work if we just choose i_0 minimal, then the minimal j_0 such that the paths P_{i_0}, P_{j_0} intersect, and then the first common point X on P_{i_0}. What could go wrong?

Proof. See Figure. In short, if we take the approach in the problem, then the involution will not be self-inverse; □

Figure is in the last page. ✓

7. (a) Show that any $n \times m$ matrix M corresponds to the path matrix between two sets $A = \{A_1, \ldots, A_n\}$ and $B = \{B_1, \ldots, B_m\}$ corresponding to the vertices of a bipartite graph G with $\mathrm{wt}(A_i \to B_j) = m_{ij}$.

(b) Use LGV to show that $\det M^T = \det M$.

(c) Given two $n \times n$ matrices M and M', use LGV to show that $\det(MM') = \det(M)\det(M')$.

(d) Let M be an $n \times p$ matrix and M' be an $p \times n$ matrix with $n \leq p$. Then

$$\det(MM') = \sum_{|R|=n} \det M_R \det M'_R,$$

where M_R is the $n \times n$ submatrix of M with columns in R and M'_R is the corresponding submatrix of M' with rows in R.

Proof. (a) Given a $n \times m$ matrix $M = (a_{ij})$, define a directed weighted graph $G_M = (V, E, wt)$ as follows:

$$V = \{A_1, \ldots, A_n, B_1, \ldots, B_m\};$$
$$E = \{e_{ij} : A_i \to B_j | 1 \leq i \leq n, 1 \leq j \leq m\},$$
$$wt(e_{ij}) = a_{ij} \text{ for all } 1 \leq i \leq n, 1 \leq j \leq m.$$

This is clearly a bipartite graph and for each choice of $(i,j) \in [n] \times [m]$, there is a unique path from A_i to B_j, namely, e_{ij}. Thus, if we set $A = \{A_1, \ldots, A_n\}$ and $B = \{B_1, \ldots, B_m\}$, then the corresponding matrix $M_{G_M} = (m_{ij})$ where

$$m_{ij} = \sum_{P: A_i \to B_j} wt(P) = wt(e_{ij}) = a_{ij}.$$

Hence, $M_{G_M} = M$, and therefore $\det M_{G_M} = \det M$.

(b) By (a), it suffices to show that $\det M_{G_{M^T}} = \det M_{G_M}$. Write $M^T = (b_{ij})$ and $M = (a_{ij})$. Then, by LGV lemma,

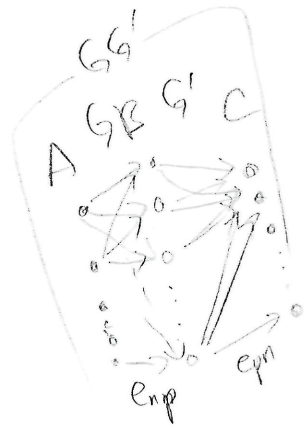
$$\det M_{G_{M^T}} = \sum_{(P_1, \ldots, P_n; \sigma)} (-1)^\sigma wt(P_1) \cdots wt(P_n) = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n b_{i\sigma(i)}.$$

(Note that no two distinct paths starting from different vertices will have a common vertice.) But, since we are summing over all permutations in S_n, we may write the last term as

$$\sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n b_{\sigma(i)i} = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n a_{i\sigma(i)} = \det M_{G_M}.$$

(c) Let M and M' be two $n \times n$ matrices. Let G_M and $G_{M'}$ be the corresponding graphs. Define the "composition" of two directed bipartite graphs G and G' to be the directed bipartite graph obtained from overlapping the target vertices of G and the source vertices of G' (respecting edges and their weights.)

More precisely, if $G = (V, E)$ with $V = \{A_1, \dots, A_n, B_1, \dots, B_l\}$ and $G' = (V', E')$ with $V' = \{B_1, \dots, B_l, C_1, \dots, C_n)$, then define GG' to be the graph having the vertex set $V_{GG'} = \{A_1, \dots, A_n, C_1, \dots, C_n\}$ with edges $E = \{e^k_{ij} | 1 \leq i, j \leq n, 1 \leq k \leq k\}$ where e^k_{ij} is the edge which begins at A_i and ends at C_j which goes through B_k. (See figure) So there are l many paths from A_i to C_j for each choice of i, j. Define $wt(e^k_{ij}) = wt(e_{ik})wt(e_{kj})$ where e_{ik} is the corresponding edge of G and e_{kj} is the corresponding edge of G'.



Now consider $M_{GG'} = (m_{ij})$. By definition, $m_{ij} = \sum_{P: A_i \to B_j} wt(P) = \sum_{k=1}^{l} wt(e^k_{ij}) = \sum_{k=1}^{l} wt(e_{ik})wt(e_{kj}) = \sum_{k=1}^{l} m^G_{ik} m^{G'}_{kj}$, where m^G_{ik} denotes the (i, k)-entry of the matrix M_{G_M}. In particular, $M_{GG'} = M_{G_M} M_{G_{M'}}$.

Now, $\det M_{GG'} = \sum_{(P_1, \dots, P, \sigma)} (-1)^\sigma \prod_{i=1}^{n} wt(P_1) \dots wt(P_n)$. By construction, there are n many paths from A_i to B_j, namely, $e^1_{ij}, \dots, e^n_{ij}$. Hence, we can write

$$\sum_{(P_1, \dots, P, \sigma)} (-1)^\sigma \prod_{i=1}^{n} wt(P_1) \dots wt(P_n) = \sum_{\sigma \in S_n} (-1)^\sigma \sum_{\tau \in S_n} wt(e^{\tau(1)}_{1\sigma(1)}) \cdots wt(e^{\tau(n)}_{n\sigma(n)}).$$

But, $wt(e^{\tau(i)}_{i\sigma(i)}) = wt(e^G_{i\tau(i)})wt(e^{G'}_{\tau(i)\sigma(i)})$. Therefore,

$$\det M_{GG'} = \sum_{\sigma, \tau} (-1)^\sigma \prod_{i=1}^{n} wt(e_{i\tau(i)})wt(e_{\tau(i)\sigma(i)})$$

$$= \sum_{\tau \in S_n} (-1)^\tau \prod_{i=1}^{n} wt(e^G_{i\tau(i)}) \sum_{\sigma \in S_n} (-1)^{\sigma\tau} \prod_{i=1}^{n} wt(e^{G'}_{\tau(i)\sigma(i)})$$

$$= \det M_{G_M} \det M_{G_{M'}},$$

where the first determinant is by definition and the second one follows from the fact the sum is over the group of permutations so that we may reindex. Therefore, $\det MM' = \det M \det M'$ by (a).

(d) In (c), we define the composition of two graph in general in the way we allow the number of overlapped vertices may be different than other set of vertices.

Let M be an $n \times p$ matrix and M' be a $p \times m$ matrix. Them as in (c), we have

$$\det M_{G_M G_{M'}} = \sum_{\sigma \in S_n} (-1)^\sigma \sum_{\tau \in S_p} \prod_{i=1}^{n} wt(e^{\tau(i)}_{i\sigma(i)})$$

$$= \sum_{\tau \in S_p} \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} wt(e_{i\sigma(i)}^{\tau(i)}).$$

Let M^τ be the $n \times n$ submatrix of M consisting of $\tau(1)$-th columns, $\dots$,$\tau(n)$-th columns and $M^{\tau\prime}$ be the $n \times n$ submatrix of M consisting of $\tau(1)$-th rows, $\dots$ $\tau(n)$-th rows. Then, we have $\det M_{G_M G_{M'}} = \sum_{\tau \in S_p} \det M^\tau \det M^{\tau\prime}$ Since τ runs over all permutations in S_p, we will be summing over all the $n \times n$ submatrices: Hence the result. ✓

$\square$

I have discussed most of the problems with Chen Chen.

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