Math 6501 - Enumerative Combinatorics I – Homework 4

Due at 3:00pm on Wednesday October 9th, 2019

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams, but individual solutions **must** be submitted for grading and credit. If you work in teams, please indicate the name of your collaborator(s) for each problem.

Problem 1. Let $F(x)$ be the ordinary generating function (o.g.f.) for the sequence $(a_n)_n$. Express the o.g.f. for the sequence $(b_n := \sum_{k=0}^{n} a_k)_n$ in terms of $F(x)$.

Problem 2. (Counting with Fibonacci Numbers)

(i) Fix $n \in \mathbb{Z}_{\geq 0}$ and consider the set $A_n := \{(a_1, \ldots, a_n) \in \{0,1\}^n : a_1 \leq a_2 \geq a_3 \leq a_4 \geq \ldots\}$. Find a close formula for $|A_n|$ in terms of Fibonacci numbers.

(ii) Fix $k, n \in \mathbb{Z}_{>0}$. Consider the set $\mathcal{T}_k := \{(T_1, \ldots, T_k) : T_i \subseteq [n] \text{ for all } i, T_1 \subseteq T_2 \supseteq T_3 \subseteq \ldots\}$. Compute $|\mathcal{T}_k|$ in terms of Fibonacci numbers.

Problem 3. Recall the Taylor series expansion $e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Show that for any $a, b \in \mathbb{C}$, the three functions e^{ax}, e^{bx} and $e^{(a+b)x}$ are well defined formal power series in x and, furthermore, $e^{(a+b)x} = e^{ax}e^{bx}$ in $\mathbb{C}[\![x]\!]$.

Problem 4. (Existence of Composition inverses)

Let $F(x) \in \mathbb{C}[\![x]\!]$ be a power series with $F(0) = 0$. Show that $F(x)$ has a composition inverse $G(x)$ in $\mathbb{C}[\![x]\!]$ (i.e. $G(x)$ with $F(G(x)) = G(F(x)) = x$) with $G(0) = 0$ if and only if $F'(0) \neq 0$.

State the corresponding equivalence where $\mathbb{C}$ is replaced by a commutative domain with unity (e.g., $\mathbb{Z}$).

Problem 5. Recall the Taylor series expansion of $\ln(1+x)$ at $x = 0$: $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$.

(i) Show that if $G(x) \in \mathbb{C}[\![x]\!]$ satisfies $G(0) = 1$, then $\ln G(x)$ is a well-defined power series over $\mathbb{C}$. Furthermore, prove that $(\ln G(x))' = G'(x)/G(x)$ as a power series.

(ii) If $G(x) = \prod_{i \geq 1} G_i(x)$ is a well-defined infinite product of non-zero series, show that $\frac{G'(x)}{G(x)} = \sum_{i \geq 1} \frac{G_i'(x)}{G_i(x)}$.

(iii) Use (ii) to compute $G'(x)/G(x)$ for $G(x) = \left(\prod_{i \geq 1}(1 - x^i)\right)^{-1}$.

Problem 6. This exercise proves that the formal power series $F(x) := \sum_{n=0}^{\infty} \binom{2n}{n} x^n$ equals $(1-4x)^{-1/2}$.

(i) Show that $a_n = \binom{2n}{n}$ satisfies $a_n = \frac{2n(2n-1)}{n^2} a_{n-1}$ for all $n \geq 1$,

(ii) Use (i) to show that the sequence $(a_n)_n$ satisfies the recursion $n\, a_n = 4n\, a_{n-1} - 2\, a_{n-1}$.

(iii) Conclude that $F(x)$ solves the differential equation $F'(x) = 4(xF(x))' - 2\,F(x)$.

(iv) Solve the differential equation to conclude that $\ln F(x) = -\frac{1}{2}\ln(1-4x)$. (*Hint:* Use Problem 5 (i).)

(v) Show that $\binom{-1/2}{n} = (\frac{-1}{4})^n \binom{2n}{n}$ and $\binom{-1}{n} = \sum_{k=0}^{n} \binom{-1/2}{k}\binom{-1/2}{n-k}$ for all $n \in \mathbb{Z}_{\geq 0}$.

(vi) Use (v) to prove that $\sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n$. Conclude that $F(x)^2 = \frac{1}{1-4x}$.

Problem 7. Determine $\sum_{n=0}^{\infty} \binom{2n+1}{n} x^n$ and $\sum_{n \geq 0} \binom{n}{\lfloor n/2 \rfloor} x^n$. (*Hint:* Use Problem 6.)