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Combinatoric 1
Homework 4

Problem 1:

$F(x)=\sum_{n=0}^{\infty} a_n x^n$, $b_n=\sum_{k=0}^{n} a_n$. Let $G(x)=\sum_{n=0}^{\infty} b_n x^n$

Then $G(x)=\dfrac{F(x)}{1-x}=F(x).(1+x+x^2+\cdots)$ ✓

Proof: $\dfrac{F(x)}{1-x}=F(x).\sum_{n=0}^{\infty}1.x^n=\left(\sum_{n=0}^{\infty}a_n x^n\right).\left(\sum_{n=0}^{\infty}1.x^n\right)$ ✓

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_n.1\right).x^n=\sum_{n=0}^{\infty}b_n x^n=G(x).$$

Problem 2:

(i) $A_0=\{\emptyset\} \Rightarrow |A_0|=1$. $A_1=\{(0),(1)\} \Rightarrow |A_1|=2$

$A_2=\{(0,0),(0,1),(1,1)\} \Rightarrow |A_2|=3$.

For $n\geq 3$:

To construct an n-tuple $(a_1 \ldots, a_n)\in A_n$, we have two options to start

1) $a_1=1$: In this case, since $(a_1 \ldots, a_n)\in A_n$, we have $1\leq a_2 \geq a_3 \leq \cdots$

So, a_2 must be also 1. For the rest we have $a_3\leq a_4\geq \cdots$

This can be chosen in $|A_{n-2}|$ many ways. ✓

2) $a_1=0$: In this case, there is no restriction on a_2. We have to determine the number of sequences: $a_2\geq a_3\leq \cdots$ $a_i=0$ or 1.

This number is the same as $|A_{n-1}|$ because there is a bijection

$\{(a_2,a_3,\ldots,a_n) \mid a_2\geq a_3\leq a_4 \cdots, a_i=0 \text{ or } 1\} \longleftrightarrow A_{n-1}$

by turning 1's into 0's & turning 0's into 1's. ✓

So, $|A_n|=|A_{n-1}|+|A_{n-2}|$ for $n\geq 3$ & $|A_2|=3=F_3$, $|A_1|=2=F_2$, $|A_0|=1=F_1$

$\Rightarrow |A_n|=F_{n+1}$ $\forall n\geq 0$. ✓

Problem 3:

$e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Assume $a, b \in \mathbb{C}$. Let $G_a(x) = ax$, $G_b(x) = bx$, $G_{(a+b)}(x) = (a+b)x \in \mathbb{C}[\![x]\!]$.
if we call $e^x = F(x) \in \mathbb{C}[\![x]\!]$, Then $e^{ax} = F(G_a(x))$, $e^{bx} = F(G_b(x))$, $e^{(a+b)x} = F(G_{(a+b)}(x))$.
Since $G_a(0) = G_b(0) = G_{(a+b)}(0) = 0$, e^{ax}, e^{bx} & $e^{(a+b)x}$ are well-defined formal power series in x. ✓

$$e^{ax} \cdot e^{bx} = \left(\sum_{n=0}^{\infty} \frac{1}{n!}(ax)^n\right) \cdot \left(\sum_{n=0}^{\infty} \frac{1}{n!}(bx)^n\right) = \left(\sum_{n=0}^{\infty} \frac{a^n}{n!} x^n\right)\left(\sum_{n=0}^{\infty} \frac{b^n}{n!} x^n\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{a^k}{k!} \cdot \frac{b^{n-k}}{(n-k)!}\right) x^n = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} n! \cdot \frac{a^k}{k!} \frac{b^{n-k}}{(n-k)!}\right) x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} a^k \cdot b^{n-k}\right) x^n = \sum_{n=0}^{\infty} \frac{1}{n!} (a+b)^n \cdot x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} [(a+b)x]^n = e^{(a+b)x}$$ ✓

$\square$.

Problem 4: $F(x) \in \mathbb{C}[\![x]\!]$ with $F(0) = 0$. So, $F(x) = \sum_{n=1}^{\infty} a_n \cdot x^n$.

Show: $F(x)$ has a composition inverse $G(x) \in \mathbb{C}[\![x]\!]$ with $G(0) = 0$

$$\Leftrightarrow$$

$$F'(0) \neq 0.$$

($\Rightarrow$) Assume $F(x)$ has a composition inverse $G(x) \in \mathbb{C}[\![x]\!]$ with $G(0) = 0$.
Say $G(x) = \sum_{n=1}^{\infty} b_n x^n$. We have $x = F(G(x)) = \sum_{n=1}^{\infty} a_n \cdot (G(x))^n$

The "x" term appears only when $n=1$ on the RHS since $G(0) = 0$. ✓
So, the coeffecient of "x" on the RHS is $a_1 \cdot b_1$
So, $\underset{\uparrow}{1} = a_1 \cdot b_1 \Rightarrow a_1 \neq 0$. $a_1 = F'(0)$. $\Rightarrow F'(0) = a_1 \neq 0$.
The coefficient of x on the LHS. ✓

Problem 5

$$\ln(1+x) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \in \mathbb{C}[[x]]$$

(i) If $G(x) \in \mathbb{C}[[x]]$ with $G(0)=1$, then $G(x) = 1 + H(x)$ with $H(0)=0$.
So, $\ln(G(x)) = \ln(1+H(x))$. If we say $F(x) = \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$, then $\ln(G(x)) = \ln(1+H(x)) = F(H(x))$. This is a well defined power series because $H(0)=0$.

To prove $\ln(G(x))' = \frac{G'(x)}{G(x)}$, let's again use $G(x) = 1+H(x)$ & $\ln(1+x) = F(x)$.

So, we want to prove that $F(H(x))' = \frac{(H(x)+1)'}{H(x)+1} = \frac{H'(x)}{H(x)+1}$

So, we want to prove $F'(H(x)) \cdot H'(x) = \frac{H'(x)}{H(x)+1}$

So, we need to show $(H(x)+1) \cdot F'(H(x)) \cdot H'(x) = H'(x)$

Hence, all we need to show is $(H(x)+1) \cdot F'(H(x)) = 1$.

First, let's see what $F'(x)$ is:

$$F'(x) = (\ln(1+x))' = \frac{d}{dx}\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n\right) = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\Rightarrow (H(x)+1) \cdot F'(H(x)) = (H(x)+1)\left(1 - H(x) + H^2(x) - H^3(x) + \cdots\right)$$

$$= [(x+1) \circ H(x)] \cdot [(1 - x + x^2 - x^3 + \cdots) \circ H(x)]$$

$$= [(x+1) \cdot (1 - x + x^2 - x^3 + \cdots)] \circ H(x)$$

$$= 1 \circ H(x)$$

$$= 1$$

$$\Rightarrow \ln(G(x))' = \frac{G'(x)}{G(x)} \quad \square$$

Here, I am using the fact that $(F \cdot G) \circ H = F(H) \cdot G(H)$ for $F, G, H \in \mathbb{C}[[x]]$ with $H(0)=0$.

(ii) $G(x) = \prod_{i=1}^{\infty} G_i(x)$. well-defined infinite product of nonzero series $G_i \in \mathbb{C}[\![x]\!]$

So, $G_i(0) = 1 \; \forall i$ & $G(0) = 1$. So, we can take $\ln$ of both sides.

$$\ln(G(x)) = \ln\left(\prod_{i=1}^{\infty} G_i(x)\right) = \ln\left(\lim_{n\to\infty} \prod_{i=1}^{n} G_i(x)\right)$$

$$= \lim_{n\to\infty} \left(\ln\left(\prod_{i=1}^{n} G_i(x)\right)\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \ln(G_i(x))$$

$$= \sum_{i=1}^{\infty} \ln(G_i(x)).$$

So, we have $\ln(G(x)) = \sum_{i=1}^{\infty} \ln(G_i(x))$. By applying $\frac{d}{dx}$ to both sides, we get:

$$\frac{G'(x)}{G(x)} = \sum_{i=1}^{\infty} \frac{G_i'(x)}{G_i(x)}$$

(Here, I am using the fact that $\frac{d}{dx}$ & limit commutes. This easily follows from fact that $f_n \to F \iff \deg(F - f_n) \to \infty$)

Problem 6:

(i) $a_n = \binom{2n}{n} = \frac{(2n)!}{n!(2n-n)!} = \frac{2n(2n-1)(2n-2)!}{n.n.(n-1)!(n-1)!} = \frac{2n(2n-1)}{n^2}\binom{2n-2}{n-1} = \frac{2n(2n-1)}{n^2} \cdot a_{n-1}$ ✓

(ii) $a_n = \frac{2\cancel{n}(2n-1)}{n^{\cancel{2}}} a_{n-1} \Rightarrow n a_n = 2(2n-1)a_{n-1} \Rightarrow n.a_n = 4n a_{n-1} - 2a_{n-1}$ ✓

(iii) $F'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n.a_n x^{n-1} = \sum_{n=1}^{\infty}(4n a_{n-1} - 2a_{n-1})x^{n-1}$

$= \sum_{n=1}^{\infty} 4n a_{n-1} x^{n-1} - \sum_{n=1}^{\infty} 2a_{n-1} x^{n-1} = 4.\sum_{n=0}^{\infty}(n+1)a_n x^n - 2.\sum_{n=0}^{\infty} a_n x^n$

$= 4(xF(x))' - 2F(x)$ ✓

(iv) $F'(x) = 4.(xF(x))' - 2F(x) = 4(F(x) + xF'(x)) - 2F(x)$

$= 4xF'(x) + 2F(x)$ ✓

$\Rightarrow F'(x)(1-4x) = 2F(x) \Rightarrow \frac{F'(x)}{F(x)} = \frac{2}{1-4x}$ integrate the equation

$\Rightarrow \ln(F(x)) = -\frac{1}{2}\ln(1-4x) + c$ ✓

c must be zero since the constant term of $\ln(F(x))$ & the constant term of $-\frac{1}{2}\ln(1-4x)$ are zero. ✓

$\Rightarrow \ln(F(x)) = -\frac{1}{2}\ln(1-4x)$

(v) $\binom{-1/2}{n} = \frac{(-1/2)(-3/2)\dots(\frac{-2n+1}{2})}{n!} = \frac{(-\frac{1}{2})^n . 1.3.5\dots(2n-1)}{n!}$ ✓

$= \left(-\frac{1}{2}\right)^n . \frac{1.3.5\dots(2n-1).2.4.6\dots 2n}{n!.2.4.6\dots.2n} = \left(-\frac{1}{2}\right)^n \frac{(2n)!}{n!.2^n.n!}$

$= \left(-\frac{1}{4}\right)^n . \frac{(2n)!}{n!n!} = \left(-\frac{1}{4}\right)^n . \binom{2n}{n}$ ✓

Before Proplem 7, let's discover some other properties of ln!

(d) $\ln : \{f \in \mathbb{C}[[x]] \mid f(0)=1\} \longrightarrow \mathbb{C}[[x]]$
$F(x) \longmapsto \ln(F(x))$ is 1-1. ✓

First, observe that $\ln(F(x))=0$ implies $F(x)=1$:
Assume $\ln(F(x))=0$. Apply $\frac{d}{dx}$ to both sides. We get

$$\frac{F'(x)}{F(x)}=0 \Rightarrow F'(x)=0 \Rightarrow F(x)=C. \text{ But } F(0)=1 \Rightarrow F(x)=1.$$ ✓

Now, assume that $\ln(F(x))=\ln(G(x))$. Then $\ln(F(x))-\ln(G(x))=0$.

$$\Rightarrow \ln\left(\frac{F(x)}{G(x)}\right)=0 \Rightarrow \frac{F(x)}{G(x)}=1 \Rightarrow F(x)=G(x)$$

$\Rightarrow \ln$ is one-to-one. ✓

(e) $\ln(F(x)^\lambda)=\ln(\underbrace{(1+H(x))^\lambda}_{H(0)=0}) = \lambda . \ln(1+H(x))=\lambda \ln(F(x))$ ✓

Just this equality have to be checked. It follows from applying $\frac{d}{dx}$ to both sides. Their derivatives turn out to be the same So, their difference is a constant. But, the constant term of $\ln((1+H(x))^\lambda)$ and the constant term of $\lambda \ln(1+H(x))$ are both zero. So, their difference is zero. So, they are the same. ✓ □

<u>Note:</u> We could have finished problem 6 at part (iv). We had

$\ln(F(x))=-\frac{1}{2}\ln(1-4x)=\ln((1-4x)^{-1/2})$ (by (e))

$\Rightarrow F(x)=(1-4x)^{-1/2}$ (by (d)) ✓

So, we have $\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^{n+1} = -\frac{1}{2}(1-4x)^{1/2} + c$

What is c?

The coefficient of x^0 on the LHS is 0.

The coefficient of x^0 on the RHS is $-\frac{1}{2}+c$.

So, $-\frac{1}{2}+c=0 \Rightarrow c=\frac{1}{2}$.

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^{n+1} = -\frac{1}{2}(1-4x)^{1/2} + \frac{1}{2}$$

$$\Rightarrow G(x) = \sum_{n=0}^{\infty}\binom{2n+1}{n} x^n = 2.F(x) - \frac{-\frac{1}{2}(1-4x)^{1/2}+\frac{1}{2}}{x}$$

$$= \boxed{2.(1-4x)^{-1/2} + \frac{(1-4x)^{1/2}-1}{2x}}$$

(As I mentioned before, x is not a unit in $\mathbb{C}[[x]]$ but x divides $(1-4x)^{1/2}-1$ so, $\frac{(1-4x)^{1/2}-1}{2x} \in \mathbb{C}[[x]]$)

Another way to solve this would be:

- Show that $(n+1)a_n = (4n+2)a_{n-1}$ $\forall n \geq 1$ where $a_n = \binom{2n+1}{n}$
- Using the recurrence relation, show that $G(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfies:

$$(x.G(x))' = 4x(xF(x))' + 2F(x) + 1$$

- Try to solve the ODE.

However, in this case, the ODE is not homogeneous but it is still doable as one can use the integrating factor method to solve.