Math 6501 - Enumerative Combinatorics I – Homework 5

Due at 3:00pm on Monday October 21st, 2019

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams, but individual solutions **must** be submitted for grading and credit. If you work in teams, please indicate the name of your collaborator(s) for each problem.

Problem 1. Consider the sequence $(a_n)_n$ defined by $a_0 = 0$, $a_1 = 1$ and $a_n = 2\,a_{n-1} + 3\,a_{n-2}$. Show that $\sum_{n=0}^{\infty} a_n\, x^n = \frac{x}{1-2x-3x^2}$, and $a_n = \frac{1}{4}\big((-1)^{n+1} + 3^n\big)$ for all $n \geq 0$.

Problem 2. (Counting lattice paths) For each $n \in \mathbb{Z}_{\geq 0}$, consider the set $\mathcal{L}_n$ of lattice paths with n steps starting from (0,0) that do not self-intersect, where we allow three types of steps: $N = (0,1)$, $E = (1,0)$ and $W = (-1,0)$. Let $f\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ be the counting function defined by $f(n) := |\mathcal{L}_n|$ for each n.

(i) Check that $f(0) = 1$ and $f(1) = 3$.

(ii) Show that for each $n \geq 2$, every path in $\mathcal{L}_n$ ends with one of the following 5 sequences (read from left to right): (1) N, (2) EE, (3) NE, (4) WW, and (5) NW.

(iii) Conclude that $f(n) = 2\,f(n-1) + f(n-2)$ for all $n \geq 2$.

(iv) Show that the o.g.f. for f equals $\frac{1+x}{1-2\,x-x^2}$ and $f(n) = \frac{1}{2}((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1})$ for all $n \geq 0$.

Problem 3. (Simultaneous Recurrences)

(i) Show that $(\sqrt{2}+\sqrt{3})^{2n} = a_n + b_n\sqrt{6}$ for all $n \in \mathbb{Z}_{\geq 0}$ for suitable sequences $(a_n)_n$ and $(b_n) \in \mathbb{Q}$.

(ii) Prove that $(a_n)_n$ and $(b_n)_n$ satisfy the simultaneous recurrences:

$$a_0 = 1, \quad b_0 = 0, \quad a_n = 5\,a_{n-1} + 12\,b_{n-1} \quad \text{and} \quad b_n = 2\,a_{n-1} + 5\,b_{n-1} \qquad \text{for all } n \geq 1$$

(iii) If $A(x)$ and $B(x)$ represent the o.g.f. for $(a_n)_n$ and $(b_n)_n$, show that

$$A(x) = 5\,x\,A(x) + 12\,x\,B(x) + 1 \quad \text{and} \quad B(x) = 2\,x\,A(x) + 5\,x\,B(x).$$

(iv) Solve for $A(x)$ and conclude that $a_n = \frac{1}{2}\big((5+2\sqrt{6})^n + (5-2\sqrt{6})^n\big)$ for $n \geq 0$. Can you determine the linear recurrence satisfied by $(a_n)_n$?

(v) Determine $B(x)$ and find the linear recurrence and close formula for $(b_n)_n$.

Problem 4. Define the *Chebyshev polynomials* by $c_0(t)=1$, $c_1(t)=t$ and $c_n(t)=t\,c_{n-1}(t)-c_{n-2}(t)$ for $n\geq 2$.

(i) Find the rational function with coefficients in $\mathbb{C}[t]$ giving the ordinary generating function of $(c_n)_n$.

(ii) Prove the explicit expression $c_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} t^{n-2k}$ for all $n \geq 0$.

Problem 5. (Evaluating partial sums) Consider a sequence $(a_n)_n$ in $\mathbb{C}$ and let $(s_n)_n$ denote the associated sequence of partial sums, i.e. $s_n := a_0 + a_1 + \ldots + a_n$ for all n. From Problem 1 in Homework 4 we know that if $A(x)$ denotes the o.g.f. for $(a_n)_n$, then $\frac{A(x)}{1-x}$ is the o.g.f. for $(s_n)_n$.

(i) Compute the o.g.f. for the *harmonic series* $(H_n)_n$ where $H_n := \sum_{j=1}^{n} \frac{1}{j}$ for all $n \geq 1$.

(ii) Show that $(H_n)_n$ satisfies the recurrence $\sum_{k=1}^{n} H_k = (n+1)(H_{n+1} - 1)$ for all $n \geq 1$. (*Hint:* Use the previous item and the binomial series for $1/(1-x)^2$.)

Problem 6. Fix $a \in \mathbb{C}$ and let $a_{m,n} := \sum_{k\geq 0} a^k \binom{m}{k}\binom{n}{k}$. Prove that $\displaystyle\sum_{m,n\geq 0} a_{m,n} y^m z^n = \frac{1}{1-y-z-(a-1)yz}$.

Problem 7. Recall the *Delannoy numbers* $D_{m,n}$ defined in Problem 7 of Homework 1, counting paths from $(0,0)$ to (m,n) using steps $(1,0)$, $(0,1)$ and $(1,1)$.

(i) Show that $D_{m,0} = D_{0,n} = 1$ for all m,n and $D_{m,n} = D_{m-1,n} + D_{m,n-1} + D_{m-1,n-1}$ for all $m,n \geq 1$.

(ii) Show that the generating function $D(y,z) := \sum_{m,n} D_{m,n}\, y^m\, z^n$ equals $\frac{1}{1-y-z-yz}$.

(iii) Use Problem 6 to conclude that $D_{m,n} = \sum_{k\geq 0} 2^k \binom{m}{k}\binom{n}{k}$ for all $m,n \geq 0$.
(*Note:* this gives a different formula for $D_{m,n}$ from the one in Homework 1).