35/35

Homework 5

Kenneth Berglund
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1. If $F(x) = \sum_{n\geq 0} a_n x^n$, then the big theorem on linear recurrences tells us that $F(x) = \frac{p(x)}{c(x)}$, where $c(x) = 1 - 2x - 3x^2$ and $p(x)$ has degree at most 1.

   Thus, $F(x) = \frac{ax+b}{1-2x-3x^2}$. Since $0 = a_0 = F(0)$, we see that $b = 0$. Also, since $1 = a_1 = F'(0)$, we see that $a = 1$ after taking a derivative. This shows that $F(x) = \frac{x}{1-2x-3x^2}$.

   The big theorem also tells us that $a_n = \sum_{i=1}^{k} p_i(n)\alpha_i^n$, where the α_i are roots of the reflection of $c(x)$ and $p_i(x)$ has degree less than the multiplicity of α_i. The roots of the reflection of $c(x)$ are 3 and (-1), so the p_i are just constants. Using $0 = a_0 = p_1 + p_2$ and $1 = a_1 = 3p_1 - p_2$, we see that $p_1 = \frac{1}{4}$ and $p_2 = -\frac{1}{4}$, so $a_n = \frac{1}{4}(3^n + (-1)^{n+1})$.

2. (i) Note that $\mathcal{L}_0 = \{()\}$, the empty path, so $f(0) = 1$. Also, $\mathcal{L}_1 = \{N, E, W\}$, so $f(1) = 3$.

   (ii) Let $P \in \mathcal{L}_n$ for $n \geq 2$. Say P does not end with the step N.

   If P ends with E, then the second-to-last step cannot be W, since then P would have a self-intersection, so the only options for endings are NE and EE. Similarly, if P ends with W, the only options for endings are NW and WW.

   This shows that all paths in $\mathcal{L}_n$ end in N, NE, EE, NW, or WW.

   (iii) We write $L_n(s_1 s_2 \dots s_k)$ for the set of paths in $\mathcal{L}_n$ which end with the sequence of steps $s_1 s_2 \dots s_k$. From Part (ii) above, $\mathcal{L}_n = L_n(N) \coprod L_n(NE) \coprod L_n(NW) \coprod L_n(EE) \coprod L_n(WW)$.

   Note that $L_n(N)$ is in bijection with $\mathcal{L}_{n-1}$, since adding an N to the end of a path in $\mathcal{L}_{n-1}$ does not introduce any self-intersections, and this is invertible, by taking away an N from a path in $L_n(N)$. Therefore, $|L_n(N)| = f(n-1)$.

   For a similar reason, $|L_n(NW)| = |L_n(NE)| = f(n-2)$, since we can add NW or NE to any path in $\mathcal{L}_{n-2}$ without worrying about introducing self-intersections.

   Now we claim $|L_n(EE) \coprod L_n(WW)| = f(n-1) - f(n-2)$. To do so, we show that $L_n(EE) \coprod L_n(WW)$ is in bijection with $\mathcal{L}_{n-1} \backslash L_{n-1}(N)$. Given a path in $L_n(EE) \coprod L_n(WW)$, we can remove the final step to obtain a path of length $n-1$ ending in E or W, that is, an element of $\mathcal{L}_{n-1} \backslash L_{n-1}(N)$. Note that repeating the last step of an element of $\mathcal{L}_{n-1} \backslash L_{n-1}(N)$ is an inverse to this process, so the sets are in bijection. Since $|\mathcal{L}_{n-1} \setminus L_{n-1}(N)| = f(n-1) - |L_{n-1}(N)| = f(n-1) - f(n-2)$, our claim is proved.

   Putting it all together, $f(n) = f(n-1) + 2f(n-2) + f(n-1) - f(n-2) = 2f(n-1) + f(n-2)$.

   (iv) Let $F(x) = \sum_{n\geq 0} f(n)x^n \in \mathbb{C}[\![x]\!]$ be the o.g.f. of $f(n)$. Using the big theorem for linear recurrences, we see that $F(x) = \frac{p(x)}{c(x)}$, where $c(x) = 1 - 2x - x^2$ and $p(x)$ has degree at most 1.

We can then write $F(x) = \frac{ax+b}{1-2x-x^2}$. Evaluating at 0 yields $1 = f(0) = F(0) = b$, and using the fact that $F'(0) = f(1) = 3$, we can take a derivative of $\frac{ax+b}{1-2x-x^2}$ and evaluate at 0 to find $a = 1$. Therefore, $F(x) = \frac{1+x}{1-2x-x^2}$.

The big theorem also tells us that $f(n) = \sum_{i=1}^{k} p_i(n)\alpha_i^n$, where the α_i are roots of the reflection of $c(x)$ and $p_i(x)$ has degree less than the multiplicity of α_i. The roots of the reflection of $c(x)$ are $1 \pm \sqrt{2}$, so the p_i are just constants. Using $1 = f(0) = p_1 + p_2$ and $3 = f(1) = p_1(1+\sqrt{2}) + p_2(1-\sqrt{2})$, we see that $p_1 = \frac{1+\sqrt{2}}{2}$ and $p_2 = \frac{1-\sqrt{2}}{2}$, so $f(n) = \frac{1}{2}((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1})$.

3. (i) We claim that for all n, $\left(\sqrt{2}+\sqrt{3}\right)^{2n} = a_n + b_n\sqrt{6}$ for some $a_n, b_n \in \mathbb{Q}$.

We proceed by induction. For $n = 0$, the value is $\left(\sqrt{2}+\sqrt{3}\right)^{2n} = 1$, so our claim holds. Now assume the claim is true for $n-1$. Then,

$$\begin{aligned}
\left(\sqrt{2}+\sqrt{3}\right)^{2n} &= \left(\sqrt{2}+\sqrt{3}\right)^{2}\left(\sqrt{2}+\sqrt{3}\right)^{2(n-1)} \\
&= \left(5+2\sqrt{6}\right)\left(a_{n-1}+b_{n-1}\sqrt{6}\right) \\
&= 5a_{n-1} + 5b_{n-1}\sqrt{6} + 2a_{n-1}\sqrt{6} + 12b_{n-1} \\
&= 5a_{n-1} + 12b_{n-1} + (5b_{n-1} + 2a_{n-1})\sqrt{6},
\end{aligned}$$

which is of the desired form. By induction, the claim is proved.

(ii) We showed above that $\left(\sqrt{2}+\sqrt{3}\right)^{0} = 1 + 0\sqrt{6}$, so $a_0 = 1$, $b_0 = 0$.
Also, our induction showed that $a_n = 5a_{n-1} + 12b_{n-1}$ and $b_n = 5b_{n-1} + 2a_{n-1}$.

(iii) Recall that for a power series $F(x) = \sum_{n\geq 0} c_n x^n$ and $\alpha \in \mathbb{C}$, the power series which has αc_{n-1} as the coefficient of x^n for $n \geq 1$ is $\alpha x F(x)$.
Applying this to the recurrence obtained in Part (ii) and accounting for the constant term, we see that $A(x) = 5xA(x) + 12xB(x) + 1$ and $B(x) = 5xB(x) + 2xA(x)$.

(iv) After performing substitution, we see that

$$A(x) = \frac{1-5x}{x^2-10x+1}.$$

The roots of the denominator are $5 \pm 2\sqrt{6}$. Using the ubiquitous big theorem, we conclude that a closed formula for $a_n = \alpha\left(5+2\sqrt{6}\right)^n + \beta\left(5-2\sqrt{6}\right)^n$ for $\alpha, \beta \in \mathbb{C}$. Using the fact that $1 = a_0 = \alpha + \beta$ and $5 = a_1 = \alpha\left(5+2\sqrt{6}\right) + \beta\left(5-2\sqrt{6}\right)$, we can solve and find that $\alpha = \beta = \frac{1}{2}$, so $a_n = \frac{1}{2}\left(\left(5+2\sqrt{6}\right)^n + \left(5-2\sqrt{6}\right)^n\right)$ for $n \geq 0$.

Also using the big theorem, we can read off the recurrence relation for a_n, since the denominator is $1 - 10x + x^2$. The recurrence relation is $a_n = 10a_{n-1} - a_{n-2}$.

(v) Using the fact that $B(x) = \frac{2xA(x)}{1-5x}$, we see that

$$B(x) = \frac{2x}{x^2-10x+1}.$$

As before, the denominator tells us that the linear recurrence for b_n is $b_n = 10b_{n-1} - b_{n-2}$. Since the roots of the denominator of $B(x)$ are the same, the closed formula is $b_n = \alpha\left(5+2\sqrt{6}\right)^n + \beta\left(5-2\sqrt{6}\right)^n$ for $\alpha, \beta \in \mathbb{C}$. Using the fact that $0 = b_0 = \alpha + \beta$ and

$2 = b_1 = \alpha\left(5+2\sqrt{6}\right) + \beta\left(5-2\sqrt{6}\right)$, we can solve and find that $\alpha = \frac{\sqrt{6}}{2\cdot 6} = -\beta$, so $b_n = \frac{\sqrt{6}}{2\cdot 6}\left(\left(5+2\sqrt{6}\right)^n - \left(5-2\sqrt{6}\right)^n\right)$.

4. (i) If $F(x) = \sum_{n\geq 0} c_n(t)x^n \in \mathbb{C}[t][[x]]$, then the big theorem on linear recurrences tells us that $F(x) = \frac{p(x)}{d(x)}$, where $d(x) = 1 - tx + x^2$ and $p(x)$ has degree at most 1.

   Writing $F(x)$ as $\frac{ax+b}{1-tx+x^2}$, we use the identity $1 = c_0(t) = F(0) = \frac{b}{1}$ to see that $b = 1$. Then we use the identity $t = c_1(t) = F'(0) = a + bt = a + t$ to find that $a = 0$. This tells us that $F(x) = \frac{1}{1-tx+x^2}$.

   (ii) We prove that $c_n(t) = \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n-k}{k}t^{n-2k}$ for all $n \geq 0$ by induction.

   For $n = 0$, $c_0(t) = 1$, and the sum also evaluates to 1.

   Now assume that $c_m(t) = \sum_{k=0}^{\lfloor m/2\rfloor}(-1)^k\binom{m-k}{k}t^{m-2k}$ for all $m < n$.

   Using the recurrence relation from Part (i), we can write

$$\begin{aligned}
c_n(t) &= tc_{n-1}(t) - c_{n-2}(t) \\
&= t\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^k\binom{n-1-k}{k}t^{n-1-2k} - \sum_{k=0}^{\lfloor (n-2)/2\rfloor}(-1)^k\binom{n-2-k}{k}t^{n-2-2k} \\
&= \sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^k\binom{n-1-k}{k}t^{n-2k} - \sum_{k=0}^{\lfloor (n-2)/2\rfloor}(-1)^k\binom{n-2-k}{k}t^{n-2-2k} \\
&= \sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^k\binom{n-1-k}{k}t^{n-2k} + \sum_{j=1}^{\lfloor (n-2)/2\rfloor+1}(-1)^j\binom{n-1-j}{j-1}t^{n-2j} \\
&= \binom{n-1}{0}t^n + \sum_{k=1}^{\lfloor (n-1)/2\rfloor}t^{n-2k}(-1)^k\left[\binom{n-1-k}{k-1} + \binom{n-1-k}{k}\right] + A(t),
\end{aligned}$$

   where $A(t)$ is nonzero if and only if n is odd. Using Pascal's recurrence, we find that this is

$$\sum_{k=0}^{\lfloor (n-1)/2\rfloor}t^{n-2k}(-1)^k\binom{n-k}{k} + A(t).$$

   If n is odd, then $\lfloor (n-1)/2\rfloor = \lfloor n/2\rfloor$, and we're done. If n is even, then

$$A(t) = (-1)^{n/2}\binom{n-1-n/2}{n/2-1}t^0 = (-1)^{n/2}\binom{n/2-1}{n/2-1} = (-1)^{n/2}\binom{n/2}{n/2},$$

   which is what we wanted.

5. (i) If $(a_n)_n$ is defined by $a_0 = 0$ and $a_n = \frac{1}{n}$ for all $n \geq 1$, then $H_n = \sum_{j=0}^{n} a_j$. Therefore, its ordinary generating function, which we denote $F(x)$, is $\frac{A(x)}{1-x}$, where $A(x)$ is the ordinary generating function for (a_n). Note that $A(x) = \sum_{n\geq 1}\frac{x^n}{n} = -\ln(1-x)$, where we used the result of Problem 5 from Homework 4.

   Therefore, the o.g.f. of (H_n) is $F(x) = \frac{-\ln(1-x)}{1-x}$.

   (ii) Another application of Problem 1 in Homework 4 tells us that $\sum_{k=1}^{n} H_k$ is the coefficient of x^n in $\frac{F(x)}{1-x} = \frac{-\ln(1-x)}{(1-x)^2}$.

Note that the o.g.f. for $(H_n - 1)_n$ is $\frac{-\ln(1-x)}{1-x} - (1+x+\cdots) = \frac{-\ln(1-x)-1}{1-x}$. If we take a derivative, we obtain the series with $(n+1)(H_{n+1}-1)$ as the coefficient of x^n. However, after applying the quotient rule, which holds for formal power series, we find that an expression for this series is $\frac{-\ln(1-x)}{(1-x)^2}$. Therefore, since their o.g.f.'s are the same, $\sum_{k=1}^{n} H_k = (n+1)(H_{n+1}-1)$.

6. We claim that $\left(\sum_{m,n\geq 0} a_{m,n} y^m z^n\right)(1-y-z-(a-1)yz) = 1$.

Note that the constant term is 1, since the only contribution is made by $a_{0,0} = \sum_{k\geq 0} a^k \binom{0}{k}\binom{0}{k} = 1$.

Now consider the coefficient of the $y^m z^n$ term. There is a contribution of $a_{m,n}$ (multiplying by 1), a contribution of $-a_{m-1,n}$ (multiplying by y), a contribution of $-a_{m,n-1}$ (multiplying by z), and a contribution of $(1-a)a_{m-1,n-1}$ (multiplying by yz). Therefore, the coefficient of $y^m z^n$ in the product is

$$
\begin{aligned}
&a_{m,n} - a_{m-1,n} - a_{m,n-1} + (1-a)a_{m-1,n-1} \\
&= \sum_{k\geq 0} a^k \left[\binom{m}{k}\binom{n}{k} - \binom{m-1}{k}\binom{n}{k} - \binom{m}{k}\binom{n-1}{k} + (1-a)\binom{m-1}{k}\binom{n-1}{k}\right] \\
&= \sum_{k\geq 0} a^k \left[\binom{n}{k}\left[\binom{m}{k} - \binom{m-1}{k}\right] - \binom{n-1}{k}\left[\binom{m}{k} - \binom{m-1}{k} + a\binom{m-1}{k}\right]\right] \\
&= \sum_{k\geq 0} a^k \left[\left[\binom{n}{k} - \binom{n-1}{k}\right]\left[\binom{m}{k} - \binom{m-1}{k}\right] - a\binom{m-1}{k}\binom{n-1}{k}\right] \\
&= \sum_{k\geq 0} a^k \left[\binom{n-1}{k-1}\binom{m-1}{k-1} - a\binom{n-1}{k}\binom{m-1}{k}\right] \\
&= \sum_{k\geq 0} a^k \binom{n-1}{k-1}\binom{m-1}{k-1} - \sum_{k\geq 0} a^{k+1}\binom{n-1}{k}\binom{m-1}{k},
\end{aligned}
$$

where we used Pascal's recurrence for the next-to-last equality. Note that the sum on the left actually starts at $k=1$, since $\binom{n-1}{-1} = \binom{m-1}{-1} = 0$, so the two sums are equal, and the coefficient of $y^m z^n$ is 0. This proves the claim.

7. (i) Let $\mathcal{D}_{m,n}$ be the set of Delannoy paths from $(0,0)$ to (m,n).

For a fixed $m \in \mathbb{Z}_{\geq 0}$, there is only one Delannoy path to $(m,0)$, namely, taking the step $(1,0)$ m times, since any other steps result in a path too high in the y direction. Thus, $D_{m,0} = 1$ for all m. Similarly, $D_{0,n} = 1$ for all n.

Now say $m, n \geq 1$. Given a Delannoy path $P \in \mathcal{D}_{m,n}$, since $m, n \geq 1$, the last step could be $(0,1)$, $(1,0)$, or $(1,1)$.

This gives rise to a bijection between $\mathcal{D}_{m,n}$ and $\mathcal{D}_{m-1,n} \coprod \mathcal{D}_{m,n-1} \coprod \mathcal{D}_{m-1,n-1}$, given by removing the last step of P. Its inverse is given by adding $(1,0)$ to a path in $\mathcal{D}_{m-1,n}$, $(0,1)$ to a path in $\mathcal{D}_{m,n-1}$, and adding $(1,1)$ to a path in $\mathcal{D}_{m-1,n-1}$.

Therefore, $D_{m,n} = D_{m-1,n} + D_{m,n-1} + D_{m-1,n-1}$ for all $m, n \geq 1$.

(ii) We claim, similar to Problem 6 above, that $\left(\sum_{m,n\geq 0} D_{m,n} y^m z^n\right)(1-y-z-yz) = 1$.

The only contribution in the product to the constant term comes from $D_{0,0} = 1$, so the constant term is 1.

The coefficient of the $y^m z^n$ term is given by $D_{m,n} - D_{m-1,n} - D_{m,n-1} - D_{m-1,n-1}$, for a similar reason to Problem 6. But by Part (i) above, this is 0, so our claim is proved.

(iii) If we let $a = 2$, Problem 6 shows that $a_{m,n} = \sum_{k\geq 0} 2^k \binom{m}{k}\binom{n}{k}$ is the coefficient of $y^m z^n$ in $\frac{1}{1-y-z-yz} = D(y,z)$. But by Part (ii), the coefficient of $y^m z^n$ in $D(y,z)$ is $D_{m,n}$, so $D_{m,n} = \sum_{k\geq 0} 2^k \binom{m}{k}\binom{n}{k}$.