Math 6501 - Enumerative Combinatorics I – Homework 6

Due at 3:00pm on Wednesday November 13th, 2019

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams, but individual solutions **must** be submitted for grading and credit. If you work in teams, please indicate the name of your collaborator(s) for each problem.

Unless otherwise indicated, compute the lattice point counting functions explicitly, i.e. **avoiding** the use of the four main statements in Ehrhart theory seen in class.

Recall: $\mathcal{L}_{\mathcal{P}}(t) := |\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^n|$, $\mathcal{L}_{\mathcal{P}^\circ}(t) := |\mathcal{P}^\circ \cap \frac{1}{t}\mathbb{Z}^n|$ for each $t \in \mathbb{Z}_{>0}$, and $\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} \mathcal{L}_{\mathcal{P}}(t) z^t$ is the Ehrhart series of a polytope $\mathcal{P}$ in $\mathbb{R}^n$.

Problem 1. Show the following formulas hold for the standard d-symplex Δ_d by explicit computations:

$$\mathcal{L}_{\Delta_d}(t) = \binom{t+d}{d}, \qquad \mathcal{L}_{\Delta_d^\circ}(t) = \binom{t-1}{d} \quad \text{and} \quad \mathrm{Ehr}_{\Delta_d}(z) = \frac{1}{(1-z)^{d+1}}.$$

Problem 2. The *Stirling numbers of the first kind* $s(n,m)$ for $m, n \in \mathbb{Z}_{\geq 0}$ (Sequence A130534 in [OEIS]) are defined as $s(0,0) = 1$, $s(0,m) = 0$ for all $m \geq 1$, and for $n, m \geq 1$ via the finite generating function:

$$x(x-1)\dots(x-n+1) = \sum_{m=0}^{n} s(n,m)\, x^m,$$

These numbers admit an alternative combinatorial interpretation: $s(n,m)$ equals $(-1)^{n-m}$ times the number of permutations of $[n]$ with exactly m cycles.

(i) Show that they satisfy the recursion: and $s(n,k) = s(n-1,k-1) - (n-1)\, s(n-1,k)$ for $n, k \geq 1$.

(ii) Show that $\frac{1}{d!}\sum_{k=0}^{d}(-1)^{d-k} s(d+1,k+1)\, t^k = \binom{t+d}{d}$. In particular, we can use $s(n,k)$ to determine the lattice-point enumerator for the standard d-simplex for each $d \in \mathbb{Z}_{>0}$.

Problem 3. (Reeve's tetrahedron) Fix $h \in \mathbb{Z}_{>0}$ and consider the Reeve's tetrahedron in $\mathbb{R}^3$ defined as

$$\mathcal{T}_h := \mathrm{conv.}(\{(0,0,0), (1,0,0), (0,1,0), (1,1,h)\}).$$

Since $\mathcal{T}_h$ is a pyramide, standard integration yields $\mathrm{vol}(\mathcal{T}_h) = h/6$.

(i) Show by direct computation that $\mathcal{L}_{\mathcal{T}_h}(1) = 4$ and $\mathcal{L}_{\mathcal{T}_h}(2) = h + 9$.

(ii) Conclude, by interpolation, that the Ehrhart polynomial equals

$$\mathcal{L}_{\mathcal{T}_h}(t) = \frac{h}{6}t^3 + t^2 + (2 - \frac{h}{6})t + 1.$$

Problem 4. The d-unit cube $\square_d$ in $\mathbb{R}^d$ is defined as the convex hull of all 2^d 0/1 vectors in $\mathbb{R}^d$.

(i) Show that its H-representation is given by $\square_d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \colon 0 \leq x_k \leq 1 \text{ for all } k = 1, \dots, d\}$.

(ii) Show that $\mathcal{L}_{\square_d}(t) = (t+1)^d$ and $\mathcal{L}_{\square_d^\circ}(t) = (t-1)^d$.

(iii) Conclude that $\mathrm{Ehr}_{\square_d}(z) = \frac{1}{z}\sum_{t \geq 1} t^d\, z^t$.

Problem 5. (Eulerian numbers) Given $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq n$, we define the *Eulerian number* $A(n,k)$ (Sequence A008292 in [OEIS]) by the generating function:

$$\sum_{j\geq 0} j^n\, z^j = \frac{\sum_{k=0}^{n} A(n,k)\, z^k}{(1-z)^{n+1}}. \tag{1}$$

Extend this definition by setting $A(n,k) = 0$ for all remaining pairs $(n,k) \in \mathbb{Z}^2$.

(i) Show by induction on n that $(1-z)^{n+1}(\sum_{j\geq 0} j^n z^j)$ is a polynomial in $\mathbb{Z}[z]$ of degree at most n. In particular, the equation (1) is valid and so the numbers $A(n,k)$ are well-defined.

(ii) Show that $\sum_{k=0}^{n} A(n,k)\, z^k$ is the numerator of the rational function $(z\frac{d}{dz})^n(\frac{1}{1-z})$. In particular, $A(n,k) \in \mathbb{Z}_{>0}$ for $0 \leq k \leq n$ and $A(n,n) = 1$ for all $n \geq 0$.

(iii) Show that $A(n,k) = A(n, n+1-k)$, $A(n,k) = (n-k+1)A(n-1,k-1) + kA(n-1,k)$ for all $n,k \geq 1$.

(iv) Conclude that $A(n,k) = \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k-j)^n$ for all $n,k \geq 0$.

(v) Use (1), and Problem 4 to conclude that $\mathrm{Ehr}_{\square_d}(z) = \dfrac{\sum_{k=1}^{d} A(d,k) z^{k-1}}{(1-z)^{d+1}}$.

Problem 6. Let $\mathcal{T}$ be the triangle with vertices $(-1/2,-1/2)$, $(1/2,-1/2)$ and $(0,3/2)$.

(i) Show that $\mathcal{L}_{\mathcal{T}}(t) = t^2 + c(t)t + 1$, where $c(t) = \begin{cases} 1 & \text{if } t \text{ is even,} \\ 0 & \text{if } t \text{ is odd.} \end{cases}$

(ii) Determine the Ehrhart series of $\mathcal{T}$.

Problem 7. (Euler's generating functions for rational polytopes)
Fix a *rational* polytope $\mathcal{P}$ in $\mathbb{R}^n_{\geq 0}$. We know that we can describe $\mathcal{P}$ by its H-representation. Furthermore, it admits the form $A\mathbf{x} \leq \mathbf{b}$ where $A \in \mathbb{Z}^{m\times n}$ and $\mathbf{b}$ is a column vector in $\mathbb{Z}^m$.

(i) Show that by adding m slack variables, we can view $\mathcal{P} \in \mathbb{R}^{n+m}_{\geq 0}$ and replace the H-representation by $A'\mathbf{x} = \mathbf{b}$ where $A' = (A|Id)$ is an $n \times (n+m)$ matrix and $\mathbf{x} \in \mathbb{R}^{n+m}_{\geq 0}$.

(ii) Show that for each $t \in \mathbb{Z}_{\geq 1}$, $t\mathcal{P} \cap \mathbb{Z}^{n+m}$ consists of those $\mathbf{x} \in \mathbb{Z}^{n+m}_{\geq 0}$ satisfying $A'\mathbf{x} = t\mathbf{b}$.

(iii) For each $t \in \mathbb{Z}_{\geq 1}$, consider the rational function

$$f(z_1, \ldots, z_m) = \frac{1}{(1-\mathbf{z}^{\mathbf{c_1}})\ldots(1-\mathbf{z}^{\mathbf{c_{m+n}}})\,\mathbf{z}^{t\mathbf{b}}},$$

where $\mathbf{c_i}$ is the i-th column of the matrix A'. Show that it can be writen as an infinite sum of Laurent monomials in $z_1, \ldots, z_m$.

(iv) Show that the Ehrhart quasipolynomial $\mathcal{L}_{\mathcal{P}}(t)$ of $\mathcal{P}$ can be computed as:

$$\mathcal{L}_{\mathcal{P}}(t) = \text{constant term of the series } f(z_1, \ldots, z_m).$$