$\frac{35}{35}$

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Math 6501 - Homework 6

1. We have that

$$\Delta_d = \{(\lambda_1, \dots, \lambda_{d+1}) \mid \lambda_1, \dots, \lambda_{d+1} \geq 0; \lambda_1 + \cdots + \lambda_{d+1} = 1\}$$

✓

so scaling by $t \in \mathbb{Z}_{>0}$ gives us

$$\begin{aligned} t\Delta_d &= \{(t\lambda_1, \dots, t\lambda_{d+1}) \mid \lambda_1, \dots, \lambda_{d+1} \geq 0; \lambda_1 + \cdots + \lambda_{d+1} = 1\} \\ &= \{(\lambda_1, \dots, \lambda_{d+1}) \mid \lambda_1, \dots, \lambda_{d+1} \geq 0; \lambda_1 + \cdots + \lambda_{d+1} = t\}. \end{aligned}$$

Intersecting with $\mathbb{Z}^{d+1}$ gives

$$t\Delta_d \cap \mathbb{Z}^{d+1} = \{(\lambda_1, \dots, \lambda_{d+1}) \mid \lambda_1, \dots, \lambda_{d+1} \in \mathbb{Z}_{\geq 0}; \lambda_1 + \cdots + \lambda_{d+1} = t\},$$

✓

which is in bijection with weak $(d+1)$-compositions of t, so

$$\mathcal{L}_{\Delta_d}(t) = \left|t\Delta_d \cap \mathbb{Z}^{d+1}\right| = \binom{t+d+1-1}{d+1-1} = \binom{t+d}{d}.$$

✓

We see that a point in the $t\Delta_d$ is on the boundary of $t\Delta_d$ exactly when one of its coordinates is 0, so

$$t\Delta_d^\circ \cap \mathbb{Z}^{d+1} = \{(\lambda_1, \dots, \lambda_{d+1}) \mid \lambda_1, \dots, \lambda_{d+1} \in \mathbb{Z}_{>0}; \lambda_1 + \cdots + \lambda_{d+1} = t\},$$

✓

which is in bijection with $(d+1)$-compositions of t, so

$$\mathcal{L}_{\Delta_d^\circ}(t) = \left|t\Delta_d^\circ \cap \mathbb{Z}^{d+1}\right| = \binom{t-1}{d+1-1} = \binom{t-1}{d}.$$

✓

Finally, we calculate

$$\begin{aligned} \mathrm{Ehr}_{\Delta_d}(z) &= 1 + \sum_{t\geq 1} \binom{t+d}{d} z^t \\ &= \sum_{t\geq 0} \binom{t+d}{d} z^t \\ &= \sum_{t\geq 0} (-1)^t \frac{(d+1)(d+2)\cdots(d+t)}{t!}(-z)^t \\ &= \sum_{t\geq 0} \frac{(-(d+1))(-(d+1)-1)\cdots(-(d+1)-t+1)}{t!}(-z)^t \\ &= \sum_{t\geq 0} \binom{-(d+1)}{t}(-z)^t \\ &= (1-z)^{-(d+1)} \\ &= \frac{1}{(1-z)^{d+1}}. \end{aligned}$$

✓

2.

(i) Suppose that $n \geq 1$. We calculate

$$\begin{aligned}
\sum_{m=0}^{n} s(n,m)x^m &= x(x-1)\cdots(x-n+1) \\
&= \left(\sum_{m=0}^{n-1} s(n-1,m)x^m\right)(x-n+1) \\
&= -(n-1)s(n-1,0) + \sum_{m=1}^{n}[s(n-1,m-1)-(n-1)s(n-1,m)]x^m. \quad \checkmark
\end{aligned}$$

The last line is from distributing, reindexing, and the recombining the coefficients for each x^m. Furthermore, we have $s(n-1,n) = 0$ so the x^n term also fits into the formula. For $k \geq 1$, we compare the coefficients of the x^k term and we get that

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k). \quad \checkmark$$

(ii) Let $f(t) = \frac{1}{d!}\sum_{k=0}^{d}(-1)^{d-k}s(d+1,k+1)t^k$. Note that $s(n,0) = 0$ since $x(x-1)\cdots(x-n+1)$ does not have a constant term.

Then our calculations yield ✓

$$\begin{aligned}
tf(t) &= \frac{1}{d!}\sum_{k=0}^{d}(-1)^{d-k}s(d+1,k+1)t^{k+1} \quad \checkmark \\
&= \frac{1}{d!}\sum_{k=1}^{d+1}(-1)^{d-k+1}s(d+1,k)t^k \\
&= \frac{1}{d!}\sum_{k=0}^{d+1}(-1)^{d-k+1}s(d+1,k)t^k \\
&= \frac{1}{d!}(-1)^{d+1}\sum_{k=0}^{d+1}(-1)^k s(d+1,k)t^k \quad \checkmark \\
&= \frac{1}{d!}(-1)^{d+1}\sum_{k=0}^{d+1} s(d+1,k)(-t)^k \\
&= \frac{1}{d!}(-1)^{d+1}(-t)(-t-1)\cdots(-t-(d+1)+1) \\
&= \frac{1}{d!}t(t+1)\cdots(t+d) \\
&= t\binom{t+d}{d}. \quad \checkmark
\end{aligned}$$

Therefore, $f(t) = \binom{t+d}{d}$, as desired. ✓

3.

(i) The pyramid $T_h \in \mathbb{R}^3_{(x,y,z)}$ is contained entirely within $\{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq x, y \leq 1; 0 \leq z \leq h\}$. The number of lattice points on T_h when $z = 0$ is 3, coming from the 3 vertices with the last coordinate 0. The intersection of the cross-section $z = s \in \mathbb{Z}$ for $0 < s < h$ and the tetrahedron is a triangle whose vertices are on the edges connecting $(1,1,h)$ and $(1,0,0)$; $(1,1,h)$ and $(0,1,0)$; and $(1,1,h)$ and $(0,0,0)$, which are defined by $(1,0,0)+(0,1,h)t$, $(0,1,0)+(1,0,h)t$, and $(0,0,0)+(1,1,h)t$, respectively, where $t \in [0,1]$. From this, we see that the vertices of the triangle cut out by the cross-section $z = s$ are $(1, s/h, s)$, $(s/h, 1, s)$, $(s/h, s/h, s)$ and since ✓ ✓

$0 < s < h$, this triangle is contained in $\operatorname{conv}\{(0,0,s),(1,0,s),(0,1,s),(1,1,s)\}$ but contains none of the lattice points of this square. Thus, these cross-sections contribute no lattice points. Finally, the $z = h$ cross-section consists of only one point and that point is a lattice point, so $\mathcal{L}_{\mathcal{T}_h}(1) = 3 + 1 = 4$.

From the calculation above, the $z = 0$ cross-section of $2\mathcal{T}_h$ has 6 lattice points, the $z = 2h$ cross-section has 1 lattice point, and the $z = s$ for $0 < s < 2h$ cross-section is

$$\operatorname{conv}\{(s/h, s/h, s), (2, s/h, s), (s/h, 2, s)\}.$$

Since this triangle cut out by the cross-section is contained in $\operatorname{conv}\{(0,0,s),(2,0,s),(0,2,s),(2,2,s)\}$, we may count the lattice points for each cross-section. For $s < h$, the point $(1,1,s)$ is the only lattice point contained in the triangle cut out by the cross-section. If $s = h$, then the three vertices of the triangle are lattice points. If $s > h$, we get no more lattice points. Thus,

$$\mathcal{L}_{\mathcal{T}_h}(2) = 6 + 1 + (h-1) + 3 = h + 9.$$

(ii) Let $s \in \mathbb{Z}$ be such that $0 \leq s \leq 3h$. We calculate similarly that

$$3\mathcal{T}_h \cap \mathbb{Z}^3 \cap \{z = s\} = \begin{cases} 10, & s = 0 \\ 3, & 0 < s < h \\ 6, & s = h \\ 1, & h < s < 2h \\ 3, & s = 2h \\ 0, & 2h < s < 3h \\ 1, & s = 3h. \end{cases}$$

Thus,

$$\mathcal{L}_{\mathcal{T}_h}(3) = 10 + 3(h-1) + 6 + 1(h-1) + 3 + 1 = 4h + 16.$$

Now let $s \in \mathbb{Z}$ be such that $0 \leq s \leq 4h$. Similarly, we have

$$4\mathcal{T}_h \cap \mathbb{Z}^3 \cap \{z = s\} = \begin{cases} 15, & s = 0 \\ 6, & 0 < s < h \\ 10, & s = h \\ 3, & h < s < 2h \\ 6, & s = 2h \\ 1, & 2h < s < 3h \\ 3, & s = 3h \\ 0, & 3h < s < 4h \\ 1, & s = 4h. \end{cases}$$

Therefore,

$$\mathcal{L}_{\mathcal{T}_h}(4) = 15 + 6(h-1) + 10 + 3(h-1) + 6 + 1(h-1) + 3 + 1 = 10h + 25.$$

We see that if $f(t) = \frac{h}{6}t^3 + t^2 + (2 - h/6)t + 1$, then $f(1) = 4$, $f(2) = h + 9$, $f(3) = 4h + 16$, $f(4) = 10h + 25$. Since $\mathcal{L}_{\mathcal{T}_h}(t)$ is also a degree 3 polynomial that agree with $f(t)$ on 4 points, we have that $f(t) - \mathcal{L}_{\mathcal{T}_h}(t)$ is a polynomial of degree at most 3 that has at least 4 zeroes, so $f(t) - \mathcal{L}_{\mathcal{T}_h}(t) = 0$ and therefore $f(t) = \mathcal{L}_{\mathcal{T}_h}(t)$, as desired.

4.

(i) The statement holds true for $d = 1$ as a base case for the induction.
Let $\{v_1, \ldots, v_{2^d}\}$ be all of the 2^d different $0/1$ vectors in $\mathbb{R}^d$. Then

$$\begin{aligned}\square_d &= \{\lambda_1 v_1 + \cdots + \lambda_{2^d} v_{2^d} \mid 0 \leq \lambda_i \leq 1, \lambda_1 + \cdots + \lambda_{2^d} = 1\} \\ &\subseteq \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 \leq x_k \leq 1 \text{ for all } k = 1, \ldots, d\}\end{aligned}$$

since each coordinate x_j of a point in $\square_d$ is a λ-weighted sum of zeroes and ones, so $0 \leq x_j \leq 1$. The reverse inclusion also holds since by induction, $\square_d \cap \{x_d = 0\} \cong \square_{d-1}$ and $\square_d \cap \{x_d = 1\} \cong \square_{d-1}$ so

$$\bigcup_{s=0}^{1} \{(x_1, \ldots, x_{d-1}, s) \in \mathbb{R}^d \mid 0 \leq x_k \leq 1 \text{ for all } k = 1, \ldots, d-1\} \subseteq \square_d.$$

Since $\square_d$ is convex, we have $\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 \leq x_k \leq 1 \text{ for all } k = 1, \ldots, d\} \subseteq \square_d$ as well, so

$$\square_d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 \leq x_k \leq 1 \text{ for all } k = 1, \ldots, d\}.$$

(ii) We have that

$$\begin{aligned}t\square_d &= \{(tx_1, \ldots, tx_d) \in \mathbb{R}^d \mid 0 \leq x_k \leq 1 \text{ for all } k = 1, \ldots, d\} \\ &= \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid 0 \leq x_k \leq t \text{ for all } k = 1, \ldots, d\}.\end{aligned}$$

Thus,

$$t\square_d \cap \mathbb{Z}^d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_k \in \mathbb{Z}, 0 \leq x_k \leq t \text{ for all } k = 1, \ldots, d\}.$$

Since each coordinate can be an integer between 0 and t independently, we have that

$$\mathcal{L}_{\square_d}(t) = (t+1)^d.$$

Similarly,

$$t\square_d^\circ \cap \mathbb{Z}^d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_k \in \mathbb{Z}, 0 < x_k < t \text{ for all } k = 1, \ldots, d\}$$

and there are $t-1$ integers strictly between 0 and t. Therefore,

$$\mathcal{L}_{\square_d^\circ}(t) = (t-1)^d.$$

(iii) By calculation

$$\begin{aligned}\mathrm{Ehr}_{\square_d}(z) &= 1 + \sum_{t \geq 1} (t+1)^d z^t \\ &= \sum_{t \geq 0} (t+1)^d z^t \\ &= \sum_{t \geq 1} (t)^d z^{t-1} \\ &= \frac{1}{z} \sum_{t \geq 1} (t)^d z^t.\end{aligned}$$

5.

(i) For $n = 0$, we have that $(1-z)^1 \sum_{j \geq 0} j^0 z^j = (1-z)\frac{1}{1-z} = 1$, which is a polynomial in $\mathbb{Z}[z]$ of degree at most n.

Now suppose $n > 0$ and the statement holds for smaller values of n. We calculate that

$$\begin{aligned}
(1-z)^{n+1}\sum_{j\geq 0} j^n z^j &= (1-z)^n \sum_{j\geq 0} j^n z^j (1-z) \\
&= (1-z)^n \left(\sum_{j\geq 0} j^n z^j - \sum_{j\geq 0} j^n z^{j+1}\right) \\
&= (1-z)^n \left(1 + \sum_{j\geq 1} (j^n - (j-1)^n) z^j\right) \\
&= (1-z)^n + (1-z)^n \sum_{j\geq 1}\sum_{k=0}^{n-1} \binom{n}{k} j^k (-1)^{n-k+1} z^j \\
&= (1-z)^n + \sum_{k=0}^{n-1} (-1)^{n-k+1}\binom{n}{k}(1-z)^{n-(k+1)}(1-z)^{k+1}\sum_{j\geq 1} j^k z^j \\
&= (1-z)^n + \sum_{k=0}^{n-1} (-1)^{n-k+1}\binom{n}{k}(1-z)^{n-(k+1)} f_k(z)
\end{aligned}$$

✓ ✓

for some polynomials $f_k(z) \in \mathbb{Z}[z]$ with degree at most k by the induction hypothesis. Then we see that the sum consists of polynomials with degree at most $n-1$ with integer coefficients, so altogether, $(1-z)^{n+1} \sum\limits_{j\geq 0} j^n z^j$ is a polynomial with integer coefficients with degree at most n. ✓

(ii) We claim that $(z\frac{d}{dz})^n(\frac{1}{1-z}) = \sum\limits_{j\geq 0} j^n z^j$. For $n = 0$, we have a geometric series and the identity holds.

Suppose that $n > 0$. Then using induction, we calculate

$$\begin{aligned}
(z\frac{d}{dz})^n(\frac{1}{1-z}) &= z\frac{d}{dz}(z\frac{d}{dz})^{n-1}(\frac{1}{1-z}) \\
&= z\frac{d}{dz}\sum_{j\geq 0} j^{n-1} z^j \\
&= z\sum_{j\geq 1} j^n z^{j-1} \\
&= \sum_{j\geq 1} j^n z^j \\
&= \sum_{j\geq 0} j^n z^j.
\end{aligned}$$

✓

This shows that the numerator of this rational function is $\sum\limits_{k=0}^{n} A(n,k) z^k$.

As for the statement that $A(n,k) \in \mathbb{Z}_{>0}$ for $0 \leq k \leq n$ and $A(n,n) = 1$ for all $n \geq 0$, we will induct on n. Suppose that $n = 0$. Then $\frac{1}{1-z}$ has a monic numerator with positive coefficients.

By induction, we suppose that

$$(z\frac{d}{dz})^{n-1}(\frac{1}{1-z}) = \frac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}}{(1-z)^n},$$

✓

where $a_0, \dots, a_{n-1} \in \mathbb{Z}_{>0}$ and $a_{n-1} = 1$. Then

$$
\begin{aligned}
(z\frac{d}{dz})^n(\frac{1}{1-z}) &= (z\frac{d}{dz})(z\frac{d}{dz})^{n-1}(\frac{1}{1-z}) \\
&= (z\frac{d}{dz})\frac{a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}}{(1-z)^n} \\
&= z\frac{(a_1+na_0)+(2a_2-a_1+na_1)z+\cdots+((n-1)a_{n-1}-(n-2)a_{n-2}+na_{n-2})z^{n-2}+(-(n-1)a_{n-1}+na_{n-1})z^{n-1}}{(1-z)^{n+1}} \\
&= \frac{(a_1+na_0)z + (2a_2+(n-1)a_1)z^2 + \cdots + ((n-1)a_{n-1}+2a_{n-2})z^{n-1} + (a_{n-1})z^n}{(1-z)^{n+1}}.
\end{aligned}
$$

We see the numerator is monic and all of the coefficients are positive, as desired.

(iii) Suppose that $n \geq 1$. From the calculation above, we have that

$$
\begin{aligned}
\sum_{k=0}^{n} A(n,k)z^k &= (a_1+na_0)z + (2a_2+(n-1)a_1)z^2 + \cdots + ((n-1)a_{n-1}+2a_{n-2})z^{n-1} + (a_{n-1})z^n \\
&= \sum_{k=1}^{n}(ka_k + (n+1-k)a_{k-1})z^k \\
&= \sum_{k=1}^{n}(kA(n-1,k) + (n+1-k)A(n-1,k-1))z^k.
\end{aligned}
$$

Comparing coefficients yields $A(n,k) = (n-k+1)A(n-1,k-1) + kA(n-1,k)$ for $k \geq 1$.

To show that $A(n,k) = A(n, n+1-k)$, we will induct on n. Consider the case where $n = 1$. For $k = 1$, we have that $A(1,1) = A(1, 1+1-1)$ and for $k = 2$, we have $A(1,2) = 0 = A(1,0)$.

Now suppose that we have $n > 1$. Then for $k \geq 1$, we have

$$
\begin{aligned}
A(n, n+1-k) &= (n-(n+1-k)+1)A(n-1, n+1-k-1) + (n+1-k)A(n-1, n+1-k) \\
&= kA(n-1,k) + (n+1-k)A(n-1, n+1-k) \\
&= kA(n-1,k) + (n+1-k)A(n-1, k-1) \\
&= A(n,k),
\end{aligned}
$$

where the equality in the third line follows from the induction hypothesis since

$$A(n-1, n-k+1) = A(n-1, n-1+1-(n-k+1)) = A(n-1, k-1).$$

(iv) For $n \geq 0$, we have that

$$
\begin{aligned}
\sum_{k=0}^{n} A(n,k)z^k &= (1-z)^{n+1}\sum_{j\geq 0} j^n z^j \\
&= \sum_{m=0}^{\infty}\binom{n+1}{m}(-1)^m z^m \sum_{j\geq 0} j^n z^j \\
&= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k-j)^n\right)z^k.
\end{aligned}
$$

Comparing coefficients of z^k yields

$$A(n,k) = \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k-j)^n$$

for $n, k \geq 0$.

(v) We have that

$$\mathrm{Ehr}_{\square_d}(z) = \frac{1}{z}\sum_{t\geq 1}(t)^d z^t = \frac{1}{z}\frac{\sum_{k=0}^{d} A(d,k)z^k}{(1-z)^{d+1}} = \frac{\sum_{k=1}^{d} A(d,k)z^{k-1}}{(1-z)^{d+1}}$$

since $A(d,0) = 0$ by (iv) when $d \geq 1$.

6.

(i) We will first calculate $\mathcal{L}_{\mathcal{T}}(t)$ for $t = 2, 4, 6$. For $t = 2$, we need to count the lattice points in $\mathrm{conv}\{(-1,-1), (1,-1), (0,3)\}$. We see that on the $x = 0$ cross-section, we have the lattice points $(0,-1), (0,0), (0,1), (0,2), (0,3)$. On the $x = 1$ cross-section, we only have one lattice point, and by symmetry, there is only one lattice point on the $x = -1$ cross-section as well. Thus, $\mathcal{L}_{\mathcal{T}}(2) = 5 + 1 + 1 = 7$.

For $t = 4$, we see the $x = 0$ cross-section has 9 lattice points. We see that the negatively sloped side of the triangle has slope -4 so the $x = 1$ cross-section is the line segment from $(1,-2)$ to $(1,2)$, so there are 5 lattice points on this cross-section. The $x = 2$ cross-section has just one lattice point. Thus, by symmetry, we have that $\mathcal{L}_{\mathcal{T}}(4) = 9 + 2(5+1) = 21$.

As for $t = 6$, we can similarly use the fact that the negatively sloped side of the triangle has slope -4 and count each cross-sections to get that $\mathcal{L}_{\mathcal{T}}(6) = 13 + 2(9+5+1) = 43$.

For the next part, we will calculate $\mathcal{L}_{\mathcal{T}}(t)$ for $t = 1, 3, 5$. By inspection, $\mathcal{L}_{\mathcal{T}}(1) = 2$. For $t = 3$, the $x = 0$ cross-section is the line segment from $(0,-3/2)$ to $(0,9/2)$, and there are 6 lattice points in between. We can use the slope of the triangle to get that the $x = 1$ cross-section is the line segment from $(1,-3/2)$ to $(1,1/2)$, which contains 2 lattice points. Thus, $\mathcal{L}_{\mathcal{T}}(3) = 6 + 2(2) = 10$.

For $t = 5$, we have the cross-sections $(0,-5/2)$ to $(0,15/2)$, $(1,-5/2)$ to $(1,7/2)$, and $(2,-5/2)$ to $(2,-1/2)$ for $x = 0,1,2$, respectively. The $x = 0$ cross-section gives 10 lattice points. The $x = 1$ cross-section gives 6 lattice points, and the $x = 2$ cross-section gives 2 lattice points. Thus, $\mathcal{L}_{\mathcal{T}}(5) = 10 + 2(6+2) = 26$.

If we restrict $\mathcal{L}_{\mathcal{T}}(t)$ to even integers, then $\mathcal{L}_{\mathcal{T}}(t)$ and t^2+t+1 are both degree 2 polynomials that agree on 3 points, so they must be the same polynomial. Similarly, $\mathcal{L}_{\mathcal{T}}(t)$ restricted to odd integers agrees with t^2+1 on 3 points, so

$$\mathcal{L}_{\mathcal{T}}(t) = t^2 + c(t)t + 1,$$

where $c(t) = 1$ if t is even and $c(t) = 0$ if t is odd.

(ii) We calculate

$$
\begin{aligned}
\mathrm{Ehr}_{\mathcal{T}}(z) &= 1 + \sum_{t=1}^{\infty} (t^2 + c(t)t + 1) z^t \\
&= \sum_{t=0}^{\infty} (t^2 + c(t)t + 1) z^t \\
&= \sum_{t=0}^{\infty} (t^2 + 1) z^t + \sum_{t=0}^{\infty} c(t) t z^t \\
&= \sum_{t=0}^{\infty} (t^2 + 1) z^t + \sum_{k=0}^{\infty} 2k z^{2k} \\
&= \sum_{t=0}^{\infty} t^2 z^t + \sum_{t=0}^{\infty} z^t + \sum_{k=0}^{\infty} 2k z^{2k} \\
&= \left(z \frac{d}{dz}\right)^2 \frac{1}{1-z} + \frac{1}{1-z} + z \frac{d}{dz} \frac{1}{1-z^2} \\
&= \frac{z + z^2}{(1-z)^3} + \frac{1}{1-z} + \frac{2z^2}{1-z^2} \\
&= \frac{1 + 2z + 4z^2 + 4z^3 + z^4 + 2z^5 + 2z^6}{(1-z^2)^3}.
\end{aligned}
$$

7.

(i) Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{P}$. Now suppose $\mathbf{y} = (y_1, \ldots, y_m) := A\mathbf{x} \le \mathbf{b} = (b_1, \ldots, b_m)$. Then $\mathbf{b} - \mathbf{y} =: (z_1, \ldots, z_m) \in \mathbb{R}^m_{\ge 0}$. We can view $\mathbf{x}$ as $(x_1, \ldots, x_n, z_1, \ldots, z_m) \in \mathbb{R}^{n+m}_{\ge 0}$. Now, $A'(x_1, \ldots, x_n, z_1, \ldots, z_m) = (y_1 + z_1, \ldots, y_m + z_m) = (b_1, \ldots, b_m) = \mathbf{b}$.

(ii) Let t be an integer such that $t \ge 1$. A point $\mathbf{x} \in \mathbb{R}^{n+m}_{\ge 0}$ is in $t\mathcal{P} \cap \mathbb{Z}^{n+m}$ if and only if $\frac{1}{t}\mathbf{x} \in \mathcal{P} \cap \frac{1}{t}\mathbb{Z}^{n+m}$. Thus, $\frac{1}{t}\mathbf{x}$ satisfies $A'(\frac{1}{t}\mathbf{x}) = \mathbf{b}$. This means that $\mathbf{x} \in \mathbb{Z}^{n+m}$ is a vector satisfying $A'\mathbf{x} = t\mathbf{b}$.

(iii) Let $i = 1, 2, \ldots, m+n$. Then $1 - \mathbf{z}^{\mathbf{c}_i}$ is an element of $\mathbb{C}[[z_1, \ldots, z_m]]$ whose constant term is a unit, so $\frac{1}{1-\mathbf{z}^{\mathbf{c}_i}} \in \mathbb{C}[[z_1, \ldots, z_m]]$ and thus

$$\frac{1}{(1-\mathbf{z}^{\mathbf{c}_1}) \cdots (1-\mathbf{z}^{\mathbf{c}_{m+n}})} \in \mathbb{C}[[z_1, \ldots, z_m]] \subseteq \mathbb{C}((z_1, \ldots, z_m)),$$

the ring of formal Laurent series in $z_1, \ldots, z_m$ over $\mathbb{C}$. Since $\frac{1}{\mathbf{z}^{t\mathbf{b}}} \in \mathbb{C}((z_1, \ldots, z_m))$, we have

$$f(z_1, \ldots, z_m) = \frac{1}{(1-\mathbf{z}^{\mathbf{c}_1}) \cdots (1-\mathbf{z}^{\mathbf{c}_{m+n}})\mathbf{z}^{t\mathbf{b}}} \in \mathbb{C}((z_1, \ldots, z_m)),$$

so $f(z_1, \ldots, z_m)$ can be written as an infinite sum of Laurent monomials in $z_1, \ldots, z_m$.

(iv) Let t to be an integer such that $t \ge 1$. We have that

$$\mathbf{z}^{t\mathbf{b}} f(z_1, \ldots, z_m) = \frac{1}{(1-\mathbf{z}^{\mathbf{c}_1}) \cdots (1-\mathbf{z}^{\mathbf{c}_{m+n}})} = (1 + \mathbf{z}^{\mathbf{c}_1} + \mathbf{z}^{2\mathbf{c}_1} + \cdots) \cdots (1 + \mathbf{z}^{\mathbf{c}_{m+n}} + \mathbf{z}^{2\mathbf{c}_{m+n}} + \cdots).$$

This means the constant term of $f(z_1, \ldots, z_m)$ is the coefficient of the $\mathbf{z}^{t\mathbf{b}}$ term of the above series. On the right hand side, we see that the coefficient of the $\mathbf{z}^{t\mathbf{b}}$ term is the cardinality of the set

$$\begin{aligned}
&\{(a_1, \dots, a_{m+n}) \in \mathbb{Z}_{\geq 0}^{m+n} \mid a_1\mathbf{c}_1 + \cdots + a_{n+m}\mathbf{c}_{n+m} = t\mathbf{b}\} \\
&= \{(a_1, \dots, a_{m+n}) \in \mathbb{Z}_{\geq 0}^{m+n} \mid A'(a_1, \dots, a_{m+n}) = t\mathbf{b}\} \\
&= t\mathcal{P} \cap \mathbb{Z}^{n+m},
\end{aligned}$$

whose cardinality is by definition $\mathcal{L}_{\mathcal{P}}(t)$. This shows that

$$\mathcal{L}_{\mathcal{P}}(t) = \text{constant term of the series } f(z_1, \dots, z_m).$$