Math 6501 - Enumerative Combinatorics I – Homework 7

Due at 3:00pm on Friday November 22th, 2019

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams, but individual solutions **must** be submitted for grading and credit. If you work in teams, please indicate the name of your collaborator(s) for each problem.

Problem 1. Computations of posets and lattices with small number of elements.

(i) Draw the Hasse diagrams of all posets with at most four elements (there are 24 of them.) Indicate which ones have $\hat{0}$, which ones have $\hat{1}$ and which ones are graded.

(ii) Draw the Hasse diagrams of all lattices with at most six elements (there are 25 of them.) Indicate which ones are graded, which ones are upper-semimodular, lower-semimodular, which ones are modular and which ones are neither.

Problem 2. Formulas for some of our favorite examples, where $\mathbf{m}$ is the m-chain poset $\{1 < 2 < \ldots < m\}$:

(i) Find a simple formula for the number of maximal chains in the partition lattice Π_n.

(ii) Prove that $B_n \simeq (\mathbf{2})^n$.

(iii) If $n = p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k}$ is the prime decomposition of the positive integer n, where p_i are pairwise distinct primes and all $m_i \in \mathbb{Z}_{>0}$, show that $D_n \simeq (\mathbf{m_1} + \mathbf{1}) \times \ldots \times (\mathbf{m_k} + \mathbf{1})$.

Problem 3. Let $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ be posets, and recall the operations used to construct new posets from old. Prove the following isomorphisms:

(i) Show that $+$ and $\times$ are associative and commutative operations.

(ii) Distributive Law: Show $\mathcal{P} \times (\mathcal{Q} + \mathcal{R}) \simeq \mathcal{P} \times \mathcal{Q} + \mathcal{P} \times \mathcal{R}$.

(iii) Show $\mathcal{P}^{\mathcal{Q}+\mathcal{R}} \simeq \mathcal{P}^{\mathcal{Q}} \times \mathcal{P}^{\mathcal{R}}$.

(iv) Show $\left(\mathcal{P}^{\mathcal{Q}}\right)^{\mathcal{R}} \simeq \mathcal{P}^{\mathcal{Q}\times\mathcal{R}}$.

Problem 4. Prove that the rank generating function $F(\mathcal{P}, q)$ for the power poset $\mathcal{P} = \mathbf{n}^{\mathbf{m}}$ obtained from the chains $\mathbf{n}$ and $\mathbf{m}$ is the q-binomial coefficient $\left[{m+n-1 \atop n-1}\right]_q$.

Problem 5. Let $\mathcal{P}$ is a finite poset and $f\colon \mathcal{P} \to \mathcal{P}$ be an order preserving bijection. Show that f is an isomorphism. Provide an example showing that this can fail when $\mathcal{P}$ is an infinite poset.

Problem 6. Let L be a lattice with meet and join operations $\wedge\colon L \times L \to L$ and $\vee\colon L \times L \to L$. Prove that these operations satisfy the following properties:

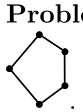
(i) associativity and commutativity,

(ii) idempotency: $s \wedge s = s \vee s = s$ for all $s \in L$,

(iii) absorption laws: $s \wedge (s \vee t) = s = s \vee (s \wedge t)$,

(iv) $s \wedge t = s$, if and only if $s \vee t = t$, if and only if $s \leq t$.

Problem 7. Show that a lattice is modular if and only if it does not contain a sublattice isomorphic to



.