33/35

M212 Burak GULEN

Combinatorics 1
Homework - 7

1)  , graded $\hat{0} = \hat{1} = \bullet$

2)  , graded, no $\hat{0}$, no $\hat{1}$

3)  , graded

4)  , graded, no $\hat{0}$, no $\hat{1}$

5)  , not graded, no $\hat{0}$, no $\hat{1}$

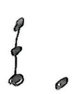
6)  , graded, no $\hat{0}$

7)  graded, no $\hat{1}$

8)  graded

9)  , graded, no $\hat{0}$, no $\hat{1}$

10)  , not graded, no $\hat{0}$, no $\hat{1}$

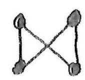
11)  , not graded, no $\hat{0}$, no $\hat{1}$

12)  , not graded, no $\hat{0}$, no $\hat{1}$

13)  , not graded, no $\hat{0}$ no $\hat{1}$

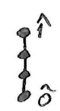
14)  , graded, no $\hat{0}$, no $\hat{1}$

15)  graded, no $\hat{0}$, no $\hat{1}$

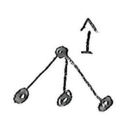
16)  graded, no $\hat{0}$, no $\hat{1}$

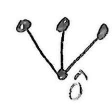
17)  , graded, no $\hat{0}$

18)  graded, no $\hat{1}$

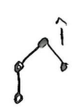
19)  graded.

20)  graded

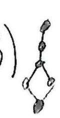
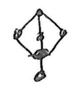
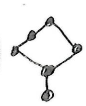
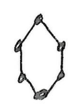
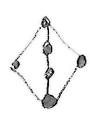
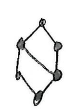
21)  graded, no $\hat{0}$

22)  graded no $\hat{1}$

23)  not graded no $\hat{1}$

24)  not graded, no $\hat{0}$.

✓ ✓

		gr	up-s-mdlr	low-s-mdlr	mdlr
1)		✓	✓	✓	✓
2)		✓	✓	✓	✓
3)		✓	✓	✓	✓
4)		✓	✓	✓	✓
5)		✓	✓	✓	✓
6)		✓	✓	✓	✓
7)		✓	✓	✓	✓
8)		✓	✓	✓	✓
9)		✓	✓	✓	✓
10)		X	X	X	X
11)		✓	✓	✓	✓
12)		✓	✓	✓	✓
13)		✓	✓	✓	✓
14)		✓	✓	✓	✓
15)		✓	✓	✓	✓
16)		✓	✓	✓	✓
17)		X	X	X	X
18)		X	X	X	X
19)		✓	✓	✓	✓
20)		✓	X	X	X
21)		X	X	X	X
22)		X	X	X	X
23)		✓	✓	✓	✓
24)		X	X	X	X
25)		X	X	X	X

Problem 2: $\underline{m} = \{1 < 2 < \dots < m\}$

(i) Let $A_n = |\{\text{maximal chains in the partition lattice } \Pi_n\}|$

To count the number of maximal chains in Π_{n+1}, we need to cover the elements which cover $\hat{0}$. Each such element determines A_n different chains ending at $\hat{1}$. So, $A_{n+1} = \binom{n}{2} . A_n$ for $n \geq 1$ & $A_1 = 1$.

→ By induction = $\frac{n!(n-1)!}{2^{n-1}}$ mn ✓

(ii) $B_n \simeq \underline{2}^n$

Define $f: B_n \to \underline{2}^n$ as $f(S) = (s_1, \dots, s_n)$ where $s_i = \begin{cases} 1 & \text{if } i \notin S \\ 2 & \text{if } i \in S \end{cases}$

f is order preserving: Assume $S \subseteq T$

So, if $i \in S$, we have $i \in T$. So, if $s_i = 2$, then $t_i = 2 \Rightarrow t_i \geq s_i$

if $i \notin S$, then $s_i = 1$. In any case ($i \in T$ or $i \notin T$) we get $t_i \geq s_i$

$\Rightarrow t_i \geq s_i$ for $i = 1, \dots, n \Rightarrow f(S) \leq f(T)$ ✓

Inverse of f: $g: \underline{2}^n \to B_n$ defined by $g(a_1, \dots, a_n) = \{i \in [n] \mid a_i = 2\}$ is the inverse of f. ✓

g is order preserving: Assume $(a_1, \dots, a_n) \leq (b_1, \dots, b_n)$

So, if $a_i = 2$, we have $b_i = 2$. So, if $i \in g(a_1, \dots, a_n)$, then $i \in g(b_1, \dots, b_n)$

$\Rightarrow g(a_1, \dots, a_n) \subseteq g(b_1, \dots, b_n)$.

✓

$\Rightarrow f$ & g are isomorphisms.

(iii) $n = p_1^{m_1} \cdots p_k^{m_k}$. Show $D_n \simeq \underline{(m_1+1)} \times \cdots \times \underline{(m_k+1)}$.

Define $\varphi: D_n \longrightarrow \underline{(m_1+1)} \times \cdots \times \underline{(m_k+1)}$

$$p_1^{n_1} \cdots p_k^{n_k} \longmapsto (n_1+1, \ldots, n_k+1)$$

<u>φ is order preserving:</u> Assume $p_1^{l_1} \cdots p_k^{l_k}$ divides $p_1^{t_1} \cdots p_k^{t_k}$

$\Rightarrow t_i \geq l_i$ for $i=1 \ldots k$ $\Rightarrow t_i+1 \geq l_i+1$ for $i=1 \ldots k$

$\Rightarrow \varphi(p_1^{l_1} \cdots p_k^{l_k}) \leq \varphi(p_1^{t_1} \cdots p_k^{t_k})$ ✓

<u>Inverse of φ:</u> $\psi: \underline{(m_1+1)} \times \cdots \times \underline{(m_k+1)} \longrightarrow D^n$ is the inverse of φ

$$(l_1, \ldots, l_k) \longmapsto p_1^{l_1-1} \cdots p_k^{l_k-1}$$ ✓

<u>ψ is order preserving:</u> Assume $(l_1, \ldots, l_k) \leq (t_1, \ldots, t_k)$

$\Rightarrow l_i - 1 \leq t_i - 1$ $\forall i=1 \ldots k$ $\Rightarrow p_1^{l_1-1} \cdots p_k^{l_k-1}$ divides $p_1^{t_1} \cdots p_k^{t_k}$

$\Rightarrow \psi(l_1, \ldots, l_k) \leq \psi(t_1, \ldots, t_k)$

✓

$\Rightarrow \varphi$ & ψ are isomorphisms.

Problem 3:

(i) $(P+Q)+R \simeq P+(Q+R)$

Since disjoint union is associative, the underlying sets are the same & identity map is the order preserving isomorphism.

$(P \times Q) \times R \simeq P \times (Q \times R)$

Since cartesian product is associative, the underlying sets are the same & identity map is the order preserving isomorphism.

For $P+Q \simeq Q+P$, identity map works again. ✓

For $P \times Q \simeq Q \times P$, the isomorphis is given by swap: $P \times Q \longrightarrow Q \times P$
$(p,q) \mapsto (q,p)$. ✓

(ii) $P \times (R+S) \simeq (P \times R) + (P \times S)$

Define $\varphi : P \times (R+S) \rightarrow (P \times R) + (P + S)$ as follows:

given $(p,t) \in P \times (R+S)$, t is either in S or R. $\Rightarrow (p,t) \in P \times R$ xor $(p,t) \in P \times S$.

So, $\varphi((p,t)) = (p,t)$. is well-defined.

Assume $(p_1, t_1) \leq (p_2, t_2) \Rightarrow p_1 \leq p_2$ & $\underbrace{t_1 \leq t_2}_{\text{either } t_1 \leq_S t_2 \text{ or } t_1 \leq_R t_2.}$ ✓

wlog assume $t_i \in S \Rightarrow p_1 \leq p_2$ & $t_1 \leq_S t_2$

$\Rightarrow \varphi((p_1, t_1)) = (p_1, t_1) \leq (p_2, t_2) = \varphi((p_2, t_2))$.

Similarly, $\psi : (P \times R) + (P + S) \longrightarrow P \times (R+S)$
$(p,t) \longmapsto (p,t)$ is a well-defined order ✓

preserving map & $\varphi \circ \psi = id$ & $\psi \circ \varphi = id$

$\Rightarrow \varphi$ & ψ are isomorphisms. ✓

(iii) $P^{Q+R} \simeq P^Q \times P^R$

An order preserving map $f: Q+R \to P$ can be seen as two order-preserving maps glued together i.e. $f = (f|_Q, f|_R)$.

So, define $\varphi: P^{Q+R} \longrightarrow P^Q \times P^R$

$$f \longmapsto (f|_Q, f|_R)$$

If $f, g \in P^{Q+R}$ with $f \geq g \Rightarrow f(q) \geq g(q)\ \forall q \in Q$ & $f(r) \geq g(r)\ \forall r \in R$

$\Rightarrow f|_Q \geq g|_Q$ & $f|_R \geq g|_R$

$\Rightarrow \varphi(f) \geq \varphi(g)$

Similarly, order-pres. maps $h: Q \to P$ & $t: R \to P$ can be glued together to determine an order-pres. map $h+t : Q+R \to P$ by

$$(h+t)(s) = \begin{cases} h(s) & \text{if } s \in Q \\ t(s) & \text{if } s \in R. \end{cases}$$

by construction we have that

$(h_1, t_1) \leq (h_2, t_2) \Rightarrow h_1 + t_1 \leq h_2 + t_2$

$\Rightarrow \psi: P^Q \times P^R \longrightarrow P^{Q+R}$ is an order-preserving map.

$$(h, t) \longmapsto h+t$$

Observe that $(h+t)|_Q = h$ & $(h+t)|_R = t$

&

$$f|_Q + f|_R = f$$

$\Rightarrow \varphi \circ \psi = id$ & $\psi \circ \varphi = id \Rightarrow \varphi$ & ψ are isomorphisms.

(iv) $(P^Q)^R \simeq P^{Q\times R}$

Define $\varphi : (P^Q)^R \longrightarrow P^{Q\times R}$

$f \longmapsto g$ where $g(q,r) = f(r)(q)$.

if $f_1 \geq f_2$ (i.e $f_1(r) \geq f_2(r)\ \forall r$)

(i.e $f_1(r)(q) \geq f_2(r)(q)\ \forall r\ \forall q$)

$\Rightarrow g_2 \geq g_1$

$\Rightarrow \varphi$ is order preserving. ✓

why does $g \in P^{Q\times R}$? (order-preserving?)

Define $\psi : P^{Q\times R} \longrightarrow (P^Q)^R$

$g \longmapsto h$ where $h(r) = g(-,r)$.

Same here, why is $h \in (P^Q)^R$ (order-pres)

If $g_1 \geq g_2$ (i.e $(q_1,r_1) \geq (q_2,r_2) \Rightarrow g(q_1,r_1) \geq g(q_2,r_2)$)

(i.e $q_1 \geq q_2 \Rightarrow g(q_1,r) \geq g(q_2,r)\ \forall r\in R$)

$\Rightarrow g_1(-,r) \geq g_2(-,r)$

$\Rightarrow \psi$ is order preserving. ✓

Also, observe that $\varphi\circ\psi = id$ & $\psi\circ\varphi = id$

$\Rightarrow \varphi$ & ψ are isomorphisms.

Problem 4: $P = \underline{n}^{\underline{m}}$. Show $F(P,q) = \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q$

Induction on $n+m$

$n+m=2 \Rightarrow P = \underline{1}^{\underline{1}} \simeq \{\cdot\} \Rightarrow F(P,q) = 1 = [1]_q = \begin{bmatrix} 1+1-1 \\ 1-1 \end{bmatrix}_q$ ✓

Assume $F(P,q) = \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q$ for $m+n=k$ where $P = n^m$.

Now assume $m'+n'=k+1$ & $P = n'^{m'}$ ✓

Observe that $P = \overbrace{\{\varphi \in P \mid \varphi(1) = 1\}}^{A} \sqcup \overbrace{\{\varphi \in P \mid \varphi(1) > 1\}}^{B}$

$\cong n'^{m'-1}$ (ignore $1 \in m'$ as it is already mapped to $1 \in n'$) ✓

$\simeq (n'-1)^{m'}$ (ignore $1 \in n'$ as nothing is mapped to $1 \in n'$) ✓

Let $\varphi_A \in A$ & let $rk_A(\varphi_A) = t$. Then, regarding $\varphi_A \in n'^{m'} = P$, we get that $rk_P(\varphi_A) = t = rk_A(\varphi_A)$ as 1 is already mapped to 1. ✓

as $A \simeq n'^{m'-1}$ we can use the formula. $(n'+m'-1 = k+1-1 = k)$

Let $\varphi_B \in B$ & let $rk_B(\varphi_B) = s$. Then regarding $\varphi_B \in n'^{m'}$, we get that $rk_P(\varphi_B) = s + m'$ as $\hat{0}_B = 2$, $\hat{0}_P = 1$, we can move m'-many terms. That is $/123\ldots m/11.1 > 1/23\ldots m/11 > 12/34\ldots m/1 > \cdots > 12\ldots m/111$

as $B \simeq (n'-1)^{m'}$, we can use the formula $(n'-1+m' = k+1-1 = k)$ ✓

$$\Rightarrow F(P,q) = \begin{bmatrix} (m'-1)+n'-1 \\ n'-1 \end{bmatrix}_q + q^{m'} \begin{bmatrix} m'+(n'-1)-1 \\ (n'-1)-1 \end{bmatrix}_q = \begin{bmatrix} m'+n'-1 \\ n'-1 \end{bmatrix}_q$$

as $rk_P(\varphi_B) = rk_B(B) + m'$ ✓

$\square$

6

Problem 5: P = finite poset $f: P \to P$ order preserving bijection.

Say $|P| = n \Rightarrow f \in S_n \Rightarrow f^n = id_P$

$\Rightarrow f^{n-1} = f^{-1} \Rightarrow f^{-1}$ is comp. of order preserving functions ✓

$\Rightarrow f^{-1}$ is order preserving

$\Rightarrow f$ is an isomorphism. ✓

Example when $|P| = \infty$ (From Stanley, Enum. Comb. Vol 1).

$P = \mathbb{Z} \cup \{t\}$ w/ usual grading on $\mathbb{Z}$ & $t \geq n \Leftrightarrow n \geq 0$. ✓

Then $f: \mathbb{Z} \cup \{t\} \to \mathbb{Z} \cup \{t\}$ is an order preserving function.

$n \mapsto n+1$ if $n \in \mathbb{Z}$

$t \mapsto t$

However $f^{-1}: P \to P$ is not order preserving

$n \mapsto n-1$

$t \mapsto t$ ✓

i.e $t \geq 0$ but $f^{-1}(t) = t \not\geq -1 = f^{-1}(0)$

Problem 6:

(i) $s \wedge t = t \wedge s$

By defn $s \wedge t \leq s$ & $s \wedge t \leq t$ & if $a \leq s, t \Rightarrow a \leq s \wedge t$

$t \wedge s \leq s$ & $t \wedge s \leq t$ & if $a \leq s, t \Rightarrow a \leq t \wedge s$ ✓

$\Rightarrow s \wedge t \leq t \wedge s$ & $t \wedge s \leq s \wedge t \Rightarrow t \wedge s = s \wedge t$.

Similarly, $t \vee s = s \vee t$. ✓ (can use duality)

$(s \wedge t) \wedge u = s \wedge (t \wedge u)$

$(s \wedge t) \wedge u \leq s, t, u \Rightarrow (s \wedge t) \wedge u \leq s, t \wedge u \Rightarrow (s \wedge t) \wedge u \leq s \wedge (t \wedge u)$

$(s) \wedge (t \wedge u) \leq s, t, u \Rightarrow s \wedge (t \wedge u) \leq (s \wedge t) \wedge u \Rightarrow s \wedge (t \wedge u) \leq (s \wedge t) \wedge u$

$\Rightarrow (s \wedge t) \wedge u = s \wedge (t \wedge u)$. Similarly $(s \vee t) \vee u = s \vee (t \vee u)$. ✓

(ii) $\underline{s \wedge s = s}$.

$$\left.\begin{array}{l} \text{By defn } s \wedge s \leq s \\ s \leq s \ \& \ s \leq s \Rightarrow s \leq s \wedge s \end{array}\right\} \Rightarrow s \wedge s = s \ \forall s \in L.$$

Similarly $s \vee s = s \ \forall s \in L$.

(iii) $\underline{s \wedge (s \vee t) = s}$

$$\left.\begin{array}{l} \text{By defn } s \wedge (s \vee t) \leq s. \\ \left.\begin{array}{l} s \leq s \vee t \\ s \leq s \end{array}\right\} \Rightarrow s \leq s \wedge (s \vee t) \end{array}\right\} \Rightarrow s \wedge (s \vee t) = s.$$

Similarly $s \vee (s \wedge t) = s$.

(iv) $\underline{s \wedge t = s \Leftrightarrow s \leq t}$

$(\Rightarrow)$ $s = s \wedge t \leq t \Rightarrow s \leq t$.

$$(\Leftarrow) \left.\begin{array}{l} \text{By defn we have } s \wedge t \leq s. \\ s \leq s \ \& \ s \leq t \Rightarrow s \leq s \wedge t \end{array}\right\} \Rightarrow s = s \wedge t$$

Similarly $s \vee t = t \Leftrightarrow s \leq t$.