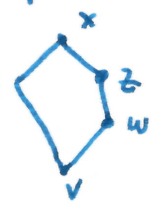
Problem 7: Assume $L$ is finite (locally finite works in the same way).

($\Rightarrow$) Prove the contrapositive.

Assume $\mathcal{P} =$ 

sublattice of $L$, so $\wedge_{\mathcal{P}} = \wedge_L$ & $\vee_{\mathcal{P}} = \vee_L$ forces

$x = y \vee z = y \vee w$ & $v = y \wedge z = y \wedge w$.

Assume $L$ is modular, so

$$\begin{array}{ccccccc} \rho(y) & + & \rho(z) & = & \rho(y\wedge z) & + & \rho(y\vee z) \\ \| & & \vee & & \| & & \| \\ \rho(y) & + & \rho(w) & = & \rho(y\wedge w) & + & \rho(y\vee w) \end{array}$$

Contra!

($\Leftarrow$) As above, we prove the contrapositive.

Assume $L$ is not modular. We want to show $L$ contains $\mathcal{P}$ as a sublattice.

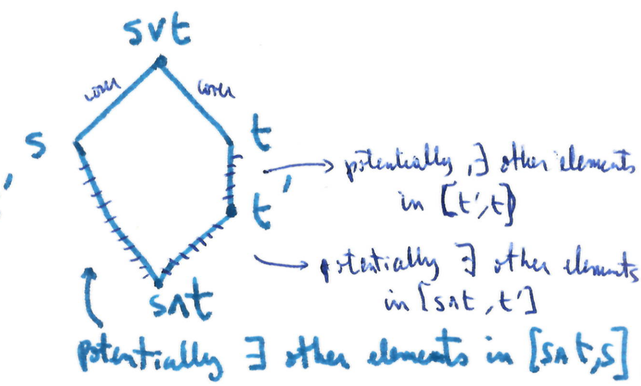
Since $\mathcal{P}^* = \mathcal{P}$, if $L$ is not modular, we may assume $L$ is not upper-semimodular

(The case where $L$ is not lower-semimodular will be obtained by dualizing)

This means $\exists\, s,t \in L$ with $s,t \lessdot s\vee t$ but $s\wedge t$ not covered by both $s$ & $t$. WLOG, assume $s \wedge t \not\lessdot t$.

So $\exists\, t' \in L$ with $s\wedge t < t' < t$

We want to show $\underset{(2)}{s\vee t' = s\vee t}$ & $\underset{(1)}{s\wedge t = s \wedge t'}$




(1) $\underset{\underset{s\wedge t}{\|}}{(s\wedge t)\wedge t'} = s\wedge(t\wedge t') = s\wedge t'$

because $s\wedge t < t'$.

(2) $t' < t \le s\vee t$ & $s \le s\vee t$ $\Rightarrow$ $s \vee t' \le s\vee t$.

Next: compare $t$ & $t'\vee s$, if possible

- If $t \le t'\vee s$ $\Rightarrow$ $s\vee t \le t'\vee s$ $\Rightarrow$ $\boxed{s\vee t' = s\vee t}$ as we wanted.

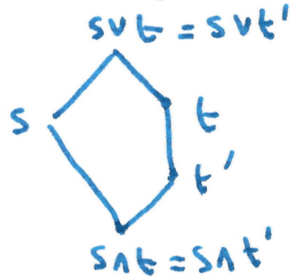
  $s \le t'\vee s$

- If $t \ge t'\vee s$ $\Longrightarrow$ $t \ge s$ contra!
- If $t$ & $t'\vee s$ are incomparable, then $t'\vee s < t\vee s$ ($\le$ & $\neq$)

But then $s \le t'\vee s < t\vee s$ & $s \lessdot t\vee s$ forces $s = t'\vee s$

But then $\left.\begin{array}{l} t' \leq t' \vee s = s \\ t' \leq t \end{array}\right\} \; t' \leq s \wedge t$ Contr!

Conclusion: of the 3 options for comparison, only one is valid and it gives $s \vee t' = s \vee t$

Conclude: 

is a sublattice of $L$.