Math 6501 - Enumerative Combinatorics I – Homework 8

Due at 3:00pm on Monday December 2nd, 2019

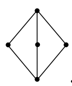
Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams, but individual solutions **must** be submitted for grading and credit. If you work in teams, please indicate the name of your collaborator(s) for each problem.

Problem 1. (Characterization of finite modular and distributive lattices)

(i) Prove that a finite lattice L is *modular* if and only if for all s, t, u in L with $s \leq u$ we have

$$s \vee (t \wedge u) = (s \vee t) \wedge u.$$

(ii) Prove that a lattice L is distributive if and only if it is modular and has no sublattice isomorphic to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Problem 2. (Distributive Laws) Let L be a lattice. Show that the following conditions are equivalent:

(i) $s \vee (t \wedge u) = (s \vee t) \wedge (s \vee u)$ for all $s, t, u \in L$.

(ii) $s \wedge (t \vee u) = (s \wedge t) \vee (s \wedge u)$ for all $s, t, u \in L$.

Problem 3. Let n be an integer, $n \geq 3$. Find the number of rank n distributive lattices such that each rank between 1 and $n-1$ (inclusive) has exactly two elements.

Problem 4. Fix $k \in \mathbb{Z}_{\geq 0}$. Given a finite distributive lattice L, we define:

- $P_k :=$ the subposet of elements that cover exactly k elements,
- $R_k :=$ the subposet of elements that are covered by exactly k elements.

Prove that $P_k \simeq R_k$ by describing an explicit isomorphism. (*Hint:* Use the fundamental theorem of finite distributive lattices.)

Problem 5. Consider the following order relation on $\mathbb{Z}^2 = \{(a, b) : a, b \in \mathbb{Z}\}$:

$$(a, b) \leq (c, d) \quad \text{if} \quad b \leq d \text{ and } |c - a| \leq d - b.$$

(i) Show that $(\mathbb{Z}^2, \leq)$ is a locally finite poset. Is it a lattice?

(ii) Find an expression for the Möbius function $\mu((0,0),(m,n))$ and show that it holds for all $m, n \in \mathbb{Z}$.

Problem 6. Suppose that L is a finite lattice.

(i) **(Crapo's lemma)**. Let X be a subset of L, and let n_k be the number of k-element subsets of X with join equal to $\hat{1}$ and meet equal to $\hat{0}$. Prove that

$$\sum_k (-1)^k n_k = -\mu(\hat{0}, \hat{1}) + \sum_{\substack{x \leq y \\ [x,y] \cap X = \emptyset}} \mu(\hat{0}, x)\mu(y, \hat{1}).$$

(ii) **(Crapo's complementation theorem)** Given any $t \in L$, prove that

$$\mu(\hat{0}, \hat{1}) = \sum_{u,v} \mu(\hat{0}, u)\zeta(u, v)\mu(v, \hat{1}),$$

where u, v range over all pairs of complements of t. Deduce that if $\mu(\hat{0}, \hat{1}) \neq 0$, then L is complemented. (*Definitions*: Given any t in L, we define a *complement* of t as an element $s \in L$ satisfying $s \vee t = \hat{1}$ and $s \wedge t = \hat{0}$. The lattice L is called *complemented* if every element has a complement.)

Problem 7. Let $(\mathcal{P}, \leq)$ be a finite poset. A map $f \colon \mathcal{P} \to \mathcal{P}$ is called a *closure operator* if for all $s, t \in \mathcal{P}$ the following three properties hold:

- $t \leq f(t)$,
- $f(f(t)) = f(t)$ (i.e., f is a projection),
- if $s \leq t$, then $f(s) \leq f(t)$.

We defined the *closed* elements of $\mathcal{P}$ as the fixed points of f, i.e., those $s \in \mathcal{P}$ with $f(s) = s$. The set $f(\mathcal{P})$ forms an induced subposet of $\mathcal{P}$. We call it the *quotient* of $\mathcal{P}$ relative to the closure operator f. We let $\mu_{f(\mathcal{P})}$ be its associated Moebius function.

(i) Prove that an element $s \in \mathcal{P}$ is closed if and only if $s \in f(\mathcal{P})$.

(ii) Show that for all $s, t \in \mathcal{P}$ we have:

$$\sum_{\substack{x \in \mathcal{P} \\ f(x) = f(t)}} \mu_{\mathcal{P}}(s, x) = \begin{cases} \mu_{f(\mathcal{P})}(f(s), f(t)) & \text{if } s = f(s), \\ 0 & \text{if } s < f(s). \end{cases}$$