MATH 6501 - ENUMERATIVE COMBINATORICS I

Lecture I: Introduction & Overview

What is Enumerative Combinatorics?

MAIN GOAL: to count! I.e., given a finite set S, or a sequence of finite sets $S_1, S_2, \ldots$, want to find $|S|$ = size of S, or $|S_1|, |S_2|, \ldots$

(*)

Trivial solution!

Caveman's Thm: $|S| = \sum_{x \in S} 1$.

⇝ . The question is not "can it be done", but rather "what does it mean to count something well, or effectively, efficiently, elegantly, etc?

. Also: what is a satisfactory answer to counting? Explicit formulas, recursions, asymptotics, generating fncs.

Def: $|S| = n$ if and only if $\exists\ S \longrightarrow \{1, 2, \ldots n\}$ bijection.

⇝ Counting through bijections.

Def: Given a sequence $(a_n)_{n \in \mathbb{N}_0}$, we have 2 basic generating functions:

(1) [OGF] Ordinary Generating Function $\sum_{n \geq 0} a_n x^n$

(2) [EGF] Exponential ——————— $\sum_{n \geq 0} a_n \frac{x^n}{n!}$

Note: These functions exist in $\mathbb{C}[[x]]$, the ring of formal power series in one variable. We will not worry about convergence.

Powers of x = placeholders & helpful notation for algebraic manipulations

Why & how to choose OGF vs EGF?

. Combinatorial Properties of Sets $\longleftrightarrow$ Algebraic Properties of Gen Functions

. EGF = better to enumerate comb structures of finite sets (eg graphs)

Three useful generating functions:

① $1 + x + x^2 + \cdots + x^i + \cdots = \sum_{i \geq 0} x^i = \frac{1}{1-x}$ Geom Series (OGF for $(1,1,\ldots)$)

② $1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{i \geq 0} \frac{x^i}{i!} = e^x$ Exp. Fnc (EGF for $(1,1,\ldots)$)

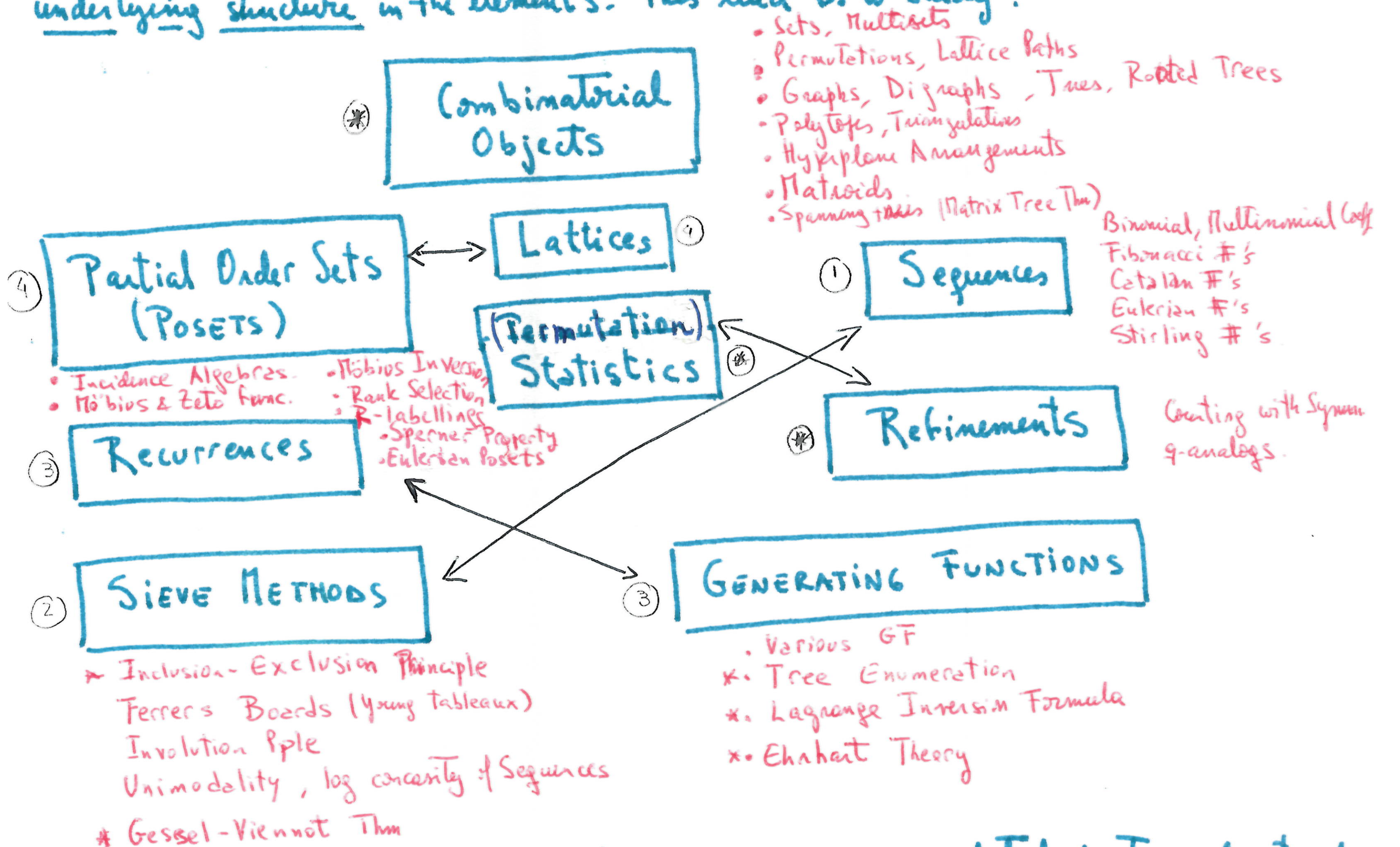
③ $0 + \frac{x^1}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots = \sum_{i=1}^{\infty} \frac{x^i}{i} = \ln\left(\frac{1}{1-x}\right) = -\ln(1-x)$

(EGF for $a_n = (n-1)!$)

Note: When series converge, we can use analytic techniques (eg $\frac{d}{dx}$, $\int$, ...).

§2 Course Overview:

General Phylosophy: To enumerate elements in a set well, we need to exploit underlying structure in the elements. This lead us to study:



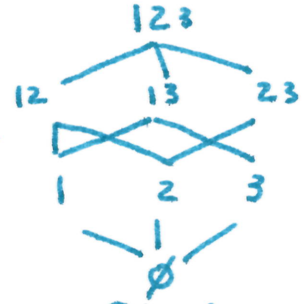
• To understand how these objects & how they are related is to understand enumeration at a basic level

• Approach:

Continuous	vs	Discrete
Gen functions	⟷	Coefficients, Recurrences, Sequences
Volume of Polytopes	⟷	Z-Vol of polytopes := $\mathrm{Vol}(P \cap \frac{1}{k}\mathbb{Z}^n)$ $[k \in \mathbb{N}]$ Ehrhart Theory
Vector Spaces Graphs	⟷	Matroids
Hyperplane Arrang	⟷	Lattices (Intersection Complex) Tutte Polynomials, Ranks, ...

§3 Examples:

Example 1: Subsets of a 3-element set $= 2^3$ of them.



$\mathcal{P}([3])$

- Partially ordered by inclusion ($A \leq B$ iff $A \subseteq B$)
- Casemen $\Rightarrow$ $2^3 = \underset{\emptyset}{1} + \underbrace{1+1+1}_{\text{singles}} + \underbrace{1+1+1}_{\text{doubles}} + \underset{\text{triple}}{1}$

  [refined count]
- Statistic: For $A \subseteq [3] = \{1,2,3\}$

  $|A| =$ size of A.

- So, 2^3 is refined by $(1,3,3,1)$
- OGF for $(1,3,3,1)$ is $1 + 3x + 3x^2 + x^3 = (1+x)^3$.

"Thm" $\underset{\substack{\text{elegant}\\ \text{efficient}}}{(1+x)^3} = \underset{\substack{|A| \subseteq [3] \\ \text{refinement} \\ \text{of Casemen}}}{\sum x^{|A|}}$ $\rightsquigarrow$ setting $x=1$ gives $2^3 = |\mathcal{P}([3])|$

Note: "Recurrences" arise when extending this to subsets of $[n] = \{1,2,3,\ldots,n\}$ $(\Leftrightarrow (2^n)_{n\geq 0})$

Example 2: Compositions.

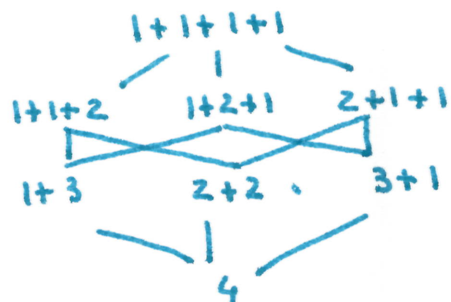
Def Given $n \in \mathbb{Z}_{\geq 0}$, a composition of n is an ordered sum $a_1 + \cdots + a_k = n$ with $a_i \in \mathbb{Z}_{>0}$.

$n=1$: 1 $\qquad 1 = 2^0$

$n=2$: 2, 1+1 $\qquad 2 = 2^1$

$n=3$: 3, 1+2, 2+1, 1+1+1 $\qquad 4 = 2^2$

$n=4$



$8 = 2^3$ (same poset as in Example 1)

Questions:
- Are there 2^{n-1} compositions of n?
- Is there a bijection between these & subsets of $[n]$ that respect a statistic inducing $(1+x)^n$ as an OGF refinement of 2^n?

- Can we find nice recurrences on these sets with combinatorial interpretations? If so, are they respected by the bijection?

  Eg $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ [Pascal Recurrence]

Answers = ALL YES

Remark The sequence $(2^n)_{n\geq 0}$ has nice OGF & EGF (with very different behavior!)

- $$\sum_{n\geq 0} 2^n x^n = \frac{1}{1-2x} \quad \text{vs} \quad \sum_{n\geq 0} \frac{2^n x^n}{n!} = e^{2x}$$

Example 3 Fibonacci Numbers

$$f_0 = 0, \quad f_1 = 1, \quad f_{k+2} = f_{k+1} + f_k \quad \text{for } k \geq 0$$

$\rightsquigarrow (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$ [OIS A000045]

OGF: $\boxed{F(x) = \sum_{k\geq 0} f_k z^k}$. (Assume convergence or work formally)

- $$\underbrace{\sum_{k\geq 0} f_{k+2} z^k}_{\frac{1}{z^2}(F(z)-z)} = \sum_{k\geq 0} (f_{k+1} + f_k) z^k = \underbrace{\sum_{k\geq 0} f_{k+1} z^k}_{\frac{1}{z}F(z)} + \underbrace{\sum_{k\geq 0} f_k z^k}_{F(z)}$$

Conclude: $\frac{F(z)-z}{z^2} = \frac{F(z)}{z} + F(z) \iff \boxed{F(z) = \frac{z}{1-z-z^2}}$

Factor denominator & use partial fraction decomposition

$$F(z) = \frac{z}{1-z-z^2} = \frac{1/\sqrt{5}}{\left(1 - \frac{1+\sqrt{5}}{2} z\right)} - \frac{1/\sqrt{5}}{\left(1 - \frac{1-\sqrt{5}}{2} z\right)} \rightsquigarrow \text{geom series!}$$

$$= \frac{1}{\sqrt{5}} \sum_{k\geq 0} \left(\frac{1+\sqrt{5}}{2} z\right)^k - \frac{1}{\sqrt{5}} \sum_{k\geq 0} \left(\frac{1-\sqrt{5}}{2} z\right)^k$$

Conclude: $\sum_{k\geq 0} f_k z^k = \sum_{k\geq 0} \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k \right) z^k$

This forces equality of coefficients $\rightsquigarrow$ Closed formula for f_k.

$$\boxed{f_k = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k \right). \quad \forall k} \quad \text{(Binet's Formula)}$$