Lecture II: Pascal's triangle & binomial coefficients

§1 Power sets & binomial coefficients

Notation: $[n] = \{1, \dots, n\}$, $2^S := \{A : A \subseteq S\}$ = power set of S,

$|S| = \# S$ = cardinality ("size") of S

GOAL: Study 2^S & recurrences arising from it.

Last time: $(|2^{[n]}|)_n = 1, 2, 4, 8, 16, \dots$

Prop 1: $|2^{[n]}| = 2^n$ for all $n \in \mathbb{N}$

Proof 1: For $T \subseteq [n-1]$, either choose to include n or not.

Thus, $\boxed{|2^{[n]}| = 2\,|2^{[n-1]}|}$ gives a recurrence $\Rightarrow$ Prove the statement by induction on n.

Proof 2 $T \in 2^{[n]}$ is determined by making a choice for each $i \in [n]$: either $i \in T$ or $i \notin T$. $\rightsquigarrow$ explicit bijection $\Theta : 2^{[n]} \longrightarrow \{0,1\}^n$

where $\Theta(T) = (\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_i = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{else} \end{cases}$

Def: Call $\Theta(T)$ = characteristic vector of A.

Observation: $2^n = (1+1)^n = \overbrace{(1+1)\,(1+1)\cdots(1+1)}^{n \text{ copies}}$

omit 1, incl 1, omit 2, incl 2, omit n, incl n

$\Rightarrow$ Argument in Proof 2 is encoded in the arithmetic, so we can turn it into algebra!

- Explicitly: $\underline{n=2}$: $(1+x_1)(1+x_2) = 1 + x_1 + x_2 + x_1x_2$

$\emptyset \quad \{1\} \quad \{2\} \quad \{1,2\}$

$\underline{n=3}$: $(1+x_1)(1+x_2)(1+x_3) = \underbrace{1 + x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3}_{\text{subsets are encoded by monomials!}}$

Prop 2: $\prod_{i=1}^{n} (1+x_i) = \sum_{k=0}^{n} \sum_{\{i_1<\dots<i_k\} \subseteq [n]} x_{i_1} \cdots x_{i_k}$

Proof: Easy induction on n.

Refinement: $2^S = \bigsqcup_{k=0}^{|S|} \underbrace{\{A \in 2^S : |A| = k\}}_{=: \binom{S}{k}}$ (disjoint!)

Name: $\binom{S}{k}$ reminds us of $\binom{n}{k}$.

If $|S| = n$, we identify it with $[n]$.

Def $\left|\binom{[n]}{k}\right| =: \binom{n}{k}$ "n choose k". (binomial coefficients)

• GOAL: Give a closed formula for $\binom{n}{k}$ $\left(= \frac{n!}{(n-k)!\,k!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} \right)$

↳ makes sense for any $n \in \mathbb{C}$!

Before: Try to see how we can learn from $\binom{n}{k}$ w/o this formula!

• **Binomial Thm**: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k =: P_n(x)$

Proof: Set $x_i = x$ for all $i = 1, \dots n$ in Prop 2. (don't need to know formula!)

Remark: Can extend this to $(y+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ using a generalization of Prop 2,

ie $\prod_{i=1}^{n} (y_i + x_i) = \sum_{A \subseteq [n]} \prod_{j \notin A} y_j \prod_{i \in A} x_i$.

§2 Pascal's Δ & comb identities:

Q: Recurrence for $\binom{n}{k}$?

• **Pascal (binomial) recurrence**: $\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$ + initial values: $\binom{n}{0} = 1$ ($\emptyset$ set)

Proof 1 (ALGEBRAIC): We know $(1+x)^n = (1+x)\underbrace{(1+x)^{n-1}}_{= P_{n-1}(x)}$ so, the Binomial Thm gives,

$$P_n(x) = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n = (1+x)\left(\binom{n-1}{0} + \binom{n-1}{1}x + \dots + \binom{n-1}{n-1}x^{n-1}\right)$$

$$\underset{\text{distribute}}{=} \binom{n-1}{0} + \left[\binom{n-1}{0} + \binom{n-1}{1}\right]x + \left[\binom{n-1}{1} + \binom{n-1}{2}\right]x^2 + \dots + \left[\binom{n-1}{n-2} + \binom{n-1}{n-1}\right]x^{n-1} + \binom{n-1}{n-1}x^n.$$

This gives two polynomials in x of degree n that coincide $\forall x$, so the coefficients on both sides agree. ∎ (Polynomial Method)

Proof 2 (COMBINATORIAL): $\binom{n}{k}$ = # of ways to pick $T \in \binom{[n]}{k}$ $= \left|\{T \in \binom{[n]}{k} \mid n \in T\} \sqcup \{T \in \binom{[n]}{k} \mid n \notin T\}\right|$

$$\underset{|A|+|B| = |A \sqcup B|}{=} \left|\{\tilde{T} \in \binom{[n-1]}{k-1}\}\right| + \left|\{\tilde{T} \in \binom{[n-1]}{k}\}\right| = \left|\binom{[n-1]}{k-1}\right| + \left|\binom{[n-1]}{k}\right| \underset{\text{def}}{=} \binom{n-1}{k-1} + \binom{n-1}{k}$$ ∎

$T = (\tilde{T} \cup \{n\})$ $\qquad$ $T = \tilde{T}$

This gives rise to **Pascals' Triangle**:

n \ k	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1			0	
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Note • $\binom{[n]}{0} = \{\emptyset\}$, $\binom{[n]}{n} = \{[n]\}$

col 1 has 1's $\qquad$ diag has 1's

• $\binom{[n]}{k} = 0$ if $k > n$.

Q: More comb identities?

Lemma 1 (Row Sum) $2^n = \sum_{k=0}^{n} \binom{n}{k}$

Proof 1: $2^{[n]} = \bigsqcup_{k=0}^{n} \binom{[n]}{k}$ & use Sum Rule for disjoint unions.
$(|A \sqcup B| = |A| + |B|)$

Proof 2: Binomial Thm with $x=1$ gives $(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k$.

Lemma 2: (Alt. Row Sum) $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 0 & \text{for } n>0 \\ 1 & \text{for } n=0 \end{cases}$

Proof Binomial Thm with $x=-1$. For $n>0$: $0^n = 0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$

(This will be the basis of Inclusion/Excl.) For $n=0$: $1 = \binom{0}{0}(-1)^0$

Lemma 3: (Symmetry) $\binom{n}{k} = \binom{n}{n-k}$

Proof $\binom{[n]}{k} \longrightarrow \binom{[n]}{n-k}$ gives explicit bijection.

$T \longmapsto [n] \setminus T$

Next Time: More binomial identities & binomial Formula.