Lecture III: More binomial identities, compositions of integers

Recall: $\binom{[n]}{k} = \{T \subseteq [n] \mid |T| = k\}$ $\qquad \binom{n}{k} := \left|\binom{[n]}{k}\right|$

Binomial Thm: $(y+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ $\qquad$ (special case: $y=1$)

Pascal's (binomial) recurrence: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ $\quad$ for all n, k

$\rightsquigarrow$ **Pascal's Triangle:**

$n \backslash k$	0	1	2	3	4	5
0	1	0	0	.	.	.
1	1	1	0	.	.	.
2	1	2	1	0	.	.
3	1	3	3	1	0	
4	1	4	6	4	1	0
5	1	5	10	10	5	1

- $\binom{n}{k} = 0$ for $k > n$
- $\binom{n}{n} = \binom{n}{0} = 1 \ \forall n$

3 Identities: $2^n = \sum_{k=0}^{n} \binom{n}{k}$ (Row sum); $\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 0 & n>0 \\ 1 & n=0 \end{cases}$ (Alt Row Sum); $\quad \binom{n}{k} = \binom{n}{n-k}$ for all $n \geq k$. (Symm.)

§1 More binomial identities:

Lemma 1: $n 2^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

Proof: Differentiate $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ & set $x = 1$.

$$n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}$$

Q: Combinatorial proof?

Lemma 2 (Partial k^{th} col sum) $\qquad \sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}$ $\quad$ for all $0 \leq k \leq n$

Proof: Classify the $(k+1)$-subsets of $[n+1]$ by the last element $= (i+1)$ (identified with $\binom{[i]}{k}$)

Lemma 3: $\sum_{i=0}^{n} \binom{m+i}{i} = \binom{m+n+1}{n}$ $\qquad$ [Diag sum from $(m,0)$ to $(m+n, n)$]

Proof: $\sum_{i=0}^{n} \binom{m+i}{i} \underset{\text{Symm}}{=} \sum_{i=0}^{n} \binom{m+i}{m} \underset{\text{1st Terms } 0}{=} \sum_{j=0}^{m+n} \binom{j}{m} \underset{\text{Lemma 3}}{=} \binom{m+n+1}{m+1} \underset{\text{Symm}}{=} \binom{m+n+1}{n}$ $\quad \square$

Vandermonde Identity: $\binom{n+m}{\ell} = \sum_{i=0}^{\ell} \binom{n}{i}\binom{m}{\ell-i}$

Proof 1: $(1+x)^{n+m} = (1+x)^n (1+x)^m$ & use Binomial Thm.

Proof 2: Pick A, B disjoint $|A| = n$, $|B| = m$. Then $T \in \binom{A \cup B}{\ell} \xleftrightarrow{\text{1-To-1}} \coprod_{i=0}^{\ell} \binom{A}{i} \times \binom{B}{\ell-i}$

$T \longmapsto (T \cap A, T \cap B)$

Theorem: $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ for $0 \le k \le n$, $k, n \in \mathbb{Z}$

Proof: We'll prove $k!\binom{n}{k} = n\cdot(n-1)\cdot\ldots\cdot(n-k+1)$.

Define $N(n,k)$ = ways of choosing $T \subseteq S$ with $|T| = k$ & then linearly ordering T.

- $\binom{n}{k}$ ways of picking T
- $k!$ ways to order linearly $\left(\begin{array}{lll} k \text{ ways to pick } 1^{st} \text{ element} \\ k-1 \;\text{———}\; 2^{nd} & " \\ \vdots \\ 1 \;\text{———}\; k^{th} & " \end{array}\right)$

So $N(n,k) = \binom{n}{k} k!$

Alternatively: $\left.\begin{array}{ll} n \text{ ways to pick } 1^{st} \text{ element of } T \\ n-1 \;\text{———}\; 2^{nd} \;\text{———}\; T \; (\text{lost } 1) \\ \vdots \\ n-(k-1) = n-k+1 \;\text{———}\; k^{th} \;\text{———}\; (\text{lost } k-1) \end{array}\right\} N(n,k) = \dfrac{n!}{(n-k)!}$

Note: This is a model for Gaussian polynomials $\begin{bmatrix} n \\ k \end{bmatrix}_q$. (q-analog of binomial coefficients).

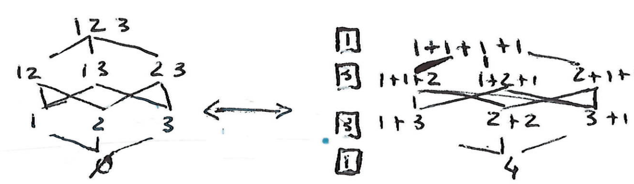
Corollary: $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ for $n, m, k \in \mathbb{Z}_{\geq 0}$ [very useful identity!]

Q: Combinatorial Proof?

- Special case $n = m$: $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$. [Vandermonde with $n = m$]

§2. Integer compositions: Fix $n > 0$ integer

In Lecture I, saw comp of $4 \longleftrightarrow 2^{[3]}$ via



Def: A *k-composition of n* is an ordered sum $a_1 + \cdots + a_k = n$ with $a_1, \ldots, a_k > 0$. Identify it with $(a_1, \ldots, a_k) \in \mathbb{Z}^k_{>0}$.

Example: $1+2+1+4+1 = 9$ is a 5-comp of 9.

Thm: There is a bijection between $\binom{[n-1]}{k-1}$ & k-compositions of n.

Corollary: $\displaystyle\sum_{k=1}^{n} |\text{k-compositions of } n| = 2^{n-1}$. (union for $1 \le k \le n$ is disjoint)

Our example above gives a method to build the bijection!

Proof of Thm: $\underline{a} = (a_1, a_2, \dots, a_k)$ with $\sum_{i=1}^{k} a_i = n$ $\xleftrightarrow{1-To-1}$ $S_{\underline{a}} = \{a_1, a_1+a_2, \dots, \sum_{i=1}^{k-1} a_i\}$ (partial sums of a_i's)

. Since $a_k > 0$, $S_{\underline{a}} \subset [n-1]$. & $|S_{\underline{a}}| = k$ because $a_1 < a_1+a_2 < \dots$

. $S_{\underline{a}}$ gives all a_i except a_k. We recover it as $a_k = n - \sum_{i=1}^{k-1} a_i$. □

Example: $(3,5,2,1) \longleftrightarrow (3,8,10) \in \binom{[10]}{3}$
4-comp of 11

Rmk: Picture for $S_{\underline{a}} \longleftrightarrow \underline{a}$ map: "stars + bars", "balls in boxes"

- Draw n dots in a row
- Draw vertical bars between $k-1$ of the $n-1$ spaces separating dots

$\cdot\cdot\cdot \mid \cdot\cdot\cdot\cdot\cdot \mid \cdot\cdot \mid \cdot$

3 5 2 1

Use 11 dots, 3 balls to left of bar 1
8 " " " " " 2
10 " " " " " 3

• Two natural extensions: $\begin{cases} \cdot \text{ allow } a_i = 0 & \text{(I)} \\ \cdot \sum_{i=1}^{k} a_i \leq n & \text{(II)} \end{cases}$

(I) **Def** A **weak k-composition** of n is an ordered sum $a_1 + \dots + a_k = n$ with $a_i \in \mathbb{Z}_{\geq 0}$
(identified with $\{(a_1, \dots, a_k) \mid a_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{k} a_i = n\}$

Ex: $1+0+2+2=5$ weak 4-comp of 5.

Prop: There are $\binom{n+k-1}{k-1}$ weak k-compositions of n.

Proof: Set $b_i = a_i + 1 \ \forall i$ $\quad (a_1, \dots, a_k) \longleftrightarrow (b_1, \dots, b_k)$.
weak k-comp of n $\quad$ k-comp of $n+k$

& $|\{k \text{ comp of } n+k\}| = \binom{n+k-1}{k-1}$. ■

Example: $1+0+2+2=5 \longleftrightarrow 2+1+3+3=9$
weak 4-comp of 5 $\quad$ 4-comp of $9=5+4$

(II) **Prop:** The # of non-neg integer solns to $\sum_{i=1}^{k} x_i \leq n$ ($x_i \geq 0 \ \forall i, x_i \in \mathbb{Z}$) is $\binom{n+k}{k}$

Proof: $\boxed{\text{"Slack Variable" Method}}$: $y = n - \sum_{i=1}^{k} x_i$ so
$(x_1, \dots, x_k, y) \in \mathbb{Z}_{\geq 0}^{k+1}$ solves $x_1 + \dots + x_k + y = n$

$\Rightarrow$ It's a weak $(k+1)$-comp of n: There are $\binom{n+k+1-1}{k+1-1} = \binom{n+k}{k}$ many! □