# Lecture IV: Multisets, multinomial coefficients

So far: • subsets of $[n]$ with fixed size ⇝ binomial coefficients
• integer (weak) compositions ⇝ we can have repeated elements!

TODAY: Subsets of $[n]$ with repetitions & multiple disjoint subsets without rep.

## §1. Multiset.

**Def**: A multiset is a set with some elements repeated a finite number of times

Ex: $M = \{\underbrace{1,1}_{\text{mult } 2}, \overset{\text{mult } 1}{2}, \underbrace{5,5,5}_{\text{mult } 3}\} = \{1^2, 2^1, 5^3\}$ (multiplicities become exponents)

Formal description: Fix a set $S$. A multiset of $S$ is a pair $(S, \nu)$, where

$\nu : S \longrightarrow \mathbb{Z}_{\geq 0}$ is the "multiplicity" function
$x \longmapsto \nu(x) = \text{mult}(x, S)$

Ex: $S = [6]$, then $M \leftrightarrow (S, \nu)$ with $\nu(1) = 2$, $\nu(2) = 1$, $\nu(3) = \nu(4) = 0$, $\nu(5) = 3$, $\nu(6) = 0$.

Names: $\sum_{x \in S} \nu(x)$ = cardinality of the multiset $M \subseteq S$ (Eg: $2+1+0+0+3+0 = 6$)

If $S = \{x_1, \ldots, x_n\}$ ⇝ $M = (S, \nu) = \{x_1^{a_1}, \ldots, x_n^{a_n}\}$ $a_i = \nu(x_i)$
$M$ is a k-multiset of $S$ if $k = \sum_{i=1}^{n} a_i$.

**Def**: $\left(\!\binom{n}{k}\!\right) :=$ # of k-multisets of $[n]$, $\left(\!\binom{[n]}{k}\!\right) = \{$k-multisets of $[n]\}$

Ex: $\left(\!\binom{3}{2}\!\right) = 6$ because $\left(\!\binom{[3]}{2}\!\right) = \{11, 22, 33, 12, 13, 23\}$

Note: We can define an order relation on multisets of $S$

$M = (S, \nu) \leq (S, \nu') = M'$ if $\nu(x) \leq \nu'(x)$ for all $x$ in $S$

(usual inclusion, but considering repetitions)

**Prop**: Given $M = (S, \nu)$, the number of submultisets of $M$ is $\prod_{x \in S} (\nu(x) + 1)$.

Pf/ • If $x \notin M$, then $\nu(x) = 0$ so $\nu(x) + 1 = 1$

• If $x \in M$, then $x$ must appear $\leq \nu(x)$ times in a submultiset ⇝ TOTAL of $\nu(x) + 1$ (include 0). □

**Theorem**: $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$

We'll given 3 proofs of this statement.

**Proof 1**: Pick $A = \{1^{a_1}, \dots, n^{a_n}\} \in \left(\!\binom{[n]}{k}\!\right)$ so $a_1 + \dots + a_n = k$, $a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}$ (weak $n$-comp of $k$)

Bijection: $\left(\!\binom{[n]}{k}\!\right) \longleftrightarrow$ weak $n$-comp of $k$

$A = \{1^{a_1}, \dots, n^{a_n}\} \longmapsto (a_1, \dots, a_n)$

Conclude $\left(\!\binom{n}{k}\!\right) = \#$ weak $n$-comp of $k$ $= \binom{n+k-1}{n-1} \underset{\text{symm}}{=} \binom{n+k-1}{n}$ □

Ex: $\{1,1,2,5,5,5\} \in \left(\!\binom{[5]}{6}\!\right) \longrightarrow (2,1,0,0,3) \in \{$weak 5-comp of 5.$\}$

**Proof 2**: Use "stars & bars": $\binom{n+k-1}{k} = \#$ of sequences of $k$ dots & $(n-1)$ vertical bars. [ $(n+k-1)$ objects & choose $k$ dots ]

$(n-1)$ bars give $n$ "boxes" or compartments $\Rightarrow v(i) = \#$ dots in $i^{th}$ box.

We can reverse this process!

Conclude $\left(\!\binom{[n]}{k}\!\right) \overset{1-1}{\longleftrightarrow}$ sequences of $k$ dots & $(n-1)$ bars. □

Ex: $\{1,1,2,5,5,5\}$ $n=5$, $k=6$ ⟿ $\bullet\bullet|\bullet|||\bullet\bullet\bullet$ $(2,1,0,0,3)$

**Proof 3**: Use technique of "compressing sequences"

$\left\{\begin{matrix}\text{strictly increasing}\\ k\text{-sequences}\\ \text{in } [n+k-1]\end{matrix}\right\}$ $a_1 < \dots < a_k$ $\xrightarrow{\text{compress}}$ $(a_i)_i \longmapsto (a_i + 1 - i)_i$ $\left\{\begin{matrix}\text{weakly increasing}\\ k\text{-sequences in } [n]\end{matrix}\right\}$ $b_1 \leq \dots \leq b_k$

expand: $(b_i + i - 1)_i \longleftarrow (b_i)_i$

Q: why $[n]$ & $[n+k-1]$? $(n+k-1 \longleftarrow n)$ largest value ← largest value

Example: $\{1 < 3 < 5 < 6 < 9\} \longmapsto (1 \leq 2 \leq 3 \leq 3 \leq 5)$

- Clearly: "compress" & "expand" are mutual inverses
- Clear:
  - $\{$weakly incr $k$-sequences in $[n]\} \overset{1-1}{\longleftrightarrow} \left(\!\binom{[n]}{k}\!\right)$
  - $\{$strictly incr. $k$-sequences in $[n+k-1]\} \overset{1-1}{\longleftrightarrow} \binom{[n+k-1]}{k}$

Conclude: $\left(\!\binom{n}{k}\!\right) = \left|\binom{[n+k-1]}{k}\right| = \binom{n+k-1}{k}$. □

Example: $\{1,1,1,1,2,2,4\} \in \left(\!\binom{[4]}{7}\!\right) \longrightarrow \{1,2,3,4,6,7,10\} \in \binom{[10]}{7}$

[weakly incr. sequence of 7 elem] [multisets can be written in weakly incr. order] — [subsets can be written in str. incr. order]

Q: Analog of Binomial Thm?

**Recall:** $(1+x_1)(1+x_2)\cdots(1+x_n)=\sum_{k=0}^{n}\sum_{i_1<\cdots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k}$ monomial $\longleftrightarrow$ subset with 0/1 exponents

To allow "repetitions", we need exponents $\geq 2$. Take:

$$(*)\quad (1+x_1+x_1^2+\cdots)(1+x_2+x_2^2+\cdots)\cdots(1+x_n+x_n^2+\cdots)=\sum_{([n],\nu)}\prod_{i=1}^{n} x_i^{\nu(i)}$$

- Each monomial $x_1^{\nu(1)}\cdots x_n^{\nu(n)}$ comes from exactly one term in each (LHS) factor
- The identity $(*)$ is viewed in $\mathbb{C}[\![x]\!]$.

**Prop 2:** $(1+x+x^2+\cdots)^n=\sum_{k\geq 0}^{\infty}\left(\!\binom{n}{k}\!\right)x^k$

**Proof:** Set $x_i=x$ for all $i$ in $(*)$ & separate $([n],\nu)$ by $\sum_{i=1}^{n}\nu(i)=k$.

**Observation:** $(1+x+x^2+\cdots)^n=\left(\frac{1}{1-x}\right)^n=(1-x)^{-n}\ \overset{\text{"}\ \text{"}}{=}\ \sum_{k\geq 0}\binom{-n}{k}(-1)^k x^k$ ⚠

gives $\left(\!\binom{n}{k}\!\right)=(-1)^k\binom{-n}{k}=(-1)^k\frac{(-n)\cdots(-n-k+1)}{k!}$ → k factors so take signs out!

$=(-1)^k(-1)^k\frac{n(n+1)\cdots(n+k-1)}{k!}=(-1)^{2k}\frac{n+k-1}{k!\,(n-1)!}=\binom{n+k-1}{k}$

Example of "combinatorial reciprocity": $\left(\!\binom{n}{k}\!\right)=(-1)^k\binom{-n}{k}$.

**Thm:** $\sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k=\frac{1}{(1-x)^n}$.

Pf/ (LHS) of Prop 2 is $\frac{1}{(1-x)^n}$ & $\left(\!\binom{n}{k}\!\right)=\binom{n+k-1}{k}$.

[because $(1-x)^n(\text{LHS})=1$ in $\mathbb{C}[\![x]\!]$ commutative ring (Exercise)]

## §2 Multinomial coefficients

**Recall:** $\binom{n}{k}$ = # of ways of selecting k objects from n to go in 1 box.

Pick 2nd box & place the rest of the $(n-k)$ elements in 2nd box.

$$T\in\binom{[n]}{k}\longleftrightarrow(T,[n]\setminus T)\longleftrightarrow[n]\setminus T\in\binom{[n]}{n-k}$$

This knows the symmetry of Pascal's Triangle!

Ex $\{12,13,23\}=\binom{[3]}{2}\longleftrightarrow\{3,2,1\}=\binom{[3]}{1}$

**Notation:** Write $\binom{n}{k}$ as $\binom{n}{k,\,n-k}$ to emphasize this. [m=2 case]

**Def:** The number of ways of placing all elements of $[n]$ into m labeled categories $C_1,\ldots,C_m$ with $a_i$ elements in $C_i$ is denoted by $\binom{n}{a_1,\ldots,a_m}$

Call it multinomial coefficient. ($a_i\in\mathbb{Z}_{\geq 0}$ & $a_1+\cdots+a_m=n$)