Lecture V: Multinomial identities, lattice paths & q-binomial coefficients
(multinomial)

Recall: $\binom{n}{a_1,\dots,a_m}$ = # ways to place all elements of $[n]$ into m labelled boxes ("categories") $C_1,\dots,C_m$ with a_i elements in C_i $\forall i$. ($a_i\in\mathbb{Z}_{\geq 0}$, $a_1+\dots+a_m=n$)

Call it multinomial coefficient.

Ex: $\binom{n}{k,\,n-k} = \binom{n}{k}$ binomial coefficient.

§1. Multinomial identities:

Thm 1: $$\binom{n}{a_1,\dots,a_m} = \prod_{j=1}^{m}\binom{n-a_1-\dots-a_{j-1}}{a_j} = \binom{n}{a_1}\binom{n-a_1}{a_2}\cdots\binom{n-a_1-\dots-a_{m-1}}{a_m} = \frac{n!}{a_1!\cdots a_m!}$$

Proof: Pick a_1 elements for C_1, then a_2 from the remaining $(n-a_1)$ for C_2, etc.

Thm 2 (Multinomial Thm): $$(x_1+\dots+x_m)^n = \sum_{\substack{a_1+\dots+a_m=n\\ a_i\geq 0}}\binom{n}{a_1,\dots,a_m}x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m}$$

Proof sketch: We need to count in how many ways we can get a fixed monomial $x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m}$. This is the same as counting in how many ways we can distribute the n factors $(x_1+\dots+x_m)$ into m categories $\mathcal{X}_1,\dots,\mathcal{X}_m$ so that we choose a_1 elements in $\mathcal{X}_1$, a_2 elements in $\mathcal{X}_2$, ..., a_m elements in $\mathcal{X}_m$ with $a_i\geq 0$, $a_1+\dots+a_m=n$. This is $\binom{n}{a_1,\dots,a_m}$ by definition! $\square$

Example $$(x_1+x_2+x_3)^4 = \underset{1}{(x_1+x_2+x_3)}\,\underset{2}{(x_1+x_2+x_3)}\,\underset{3}{(x_1+x_2+x_3)}\,\underset{4}{(x_1+x_2+x_3)}$$

$x_2\cdot x_3\cdot x_1\cdot x_1$ choice corresponds to $\underset{\mathcal{X}_1}{\{3,4\}},\ \underset{\mathcal{X}_2}{\{1\}}\ \underset{\mathcal{X}_3}{\{2\}}$.

Thm 3 (Permutations of a multiset)

$M=\{x_1^{a_1},\dots,x_m^{a_m}\}=(S,\nu)$ multiset with $\sum_{i=1}^m a_i = n$. Then defining $G_M :=$ permutations of M, gives $|G_M| = \binom{n}{a_1,\dots,a_m}$

- How to interpret G_M?

G_S = permutations of set S = $G_{|S|}$ or $\mathfrak{S}_{|S|}$.

= linear ordering $w_1,\dots,w_{|S|}$ of the elements of S (viewed as a word $w_1\cdots w_{|S|}$)

= bijections $S\to S$

$x_i \to w_i$

Know $|G_S| = (|S|)! = \underset{\text{choices for } w_1}{|S|}\ \underset{w_2}{(|S|-1)}\cdots\underset{w_{|S|}}{(|S|-|S|+1)}$

Def $\mathfrak{S}_M$ = linear ordering of the "elements" of M.

$M = (S, \nu)$ $\rightsquigarrow$ each $x \in S$ appears exactly $\nu(x)$ times in the permutation

Easier: Think of ordering as a <u>word</u> $w_1 \cdots\cdots w_n$ $\quad (n = |M|)$

Example $\mathfrak{S}_{1123} = \{1123, 1132, 1213, 1312, 1231, 1321, 2113, 2131, 3112, 3121, 3211\}$ $\rightsquigarrow |\mathfrak{S}_{1123}| = 12 = \binom{4}{2,1,1}$

Proof of Thm 3: If x_i appears in position j of $\sigma \in \mathfrak{S}_M$ $(w_j = i)$, then, place $j \in [n]$ in $\mathfrak{X}_i$. (m categories $\mathfrak{X}_1 \leftrightarrow x_1, \ldots, \mathfrak{X}_m \leftrightarrow x_m$

Place "locations" of x_i in a word into $\mathfrak{X}_i$)

Example: $M = \{1,1,1,2,2,3,3\}$ & $\sigma = \overset{1}{2}\,\overset{2}{1}\,\overset{3}{3}\,\overset{4}{1}\,\overset{5}{1}\,\overset{6}{3}\,\overset{7}{2}$ corresponds to $\underbrace{2,4,5}_{\mathfrak{X}_1}, \underbrace{1,7}_{\mathfrak{X}_2}, \underbrace{3,6}_{\mathfrak{X}_3}$.

$n = 7$

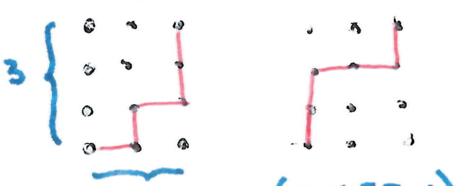
§ 2. Lattice paths:

$S \subseteq \mathbb{Z}^d$ $\quad d \geq 2$ $\quad \{e_1, \ldots, e_n\}$ standard basis of $\mathbb{Z}^d$, $e_i = (0, \ldots 0, \overset{i}{1}, 0 \cdots 0)$

Def: A <u>lattice path</u> L in $\mathbb{Z}^d$ of length k with steps in S is a sequence $v_0, v_1, \ldots, v_k \in \mathbb{Z}^d$ such that each consecutive difference $v_i - v_{i-1}$ lies in S

We say L starts at v_0 and ends at v_k, or goes from v_0 to v_k.

Example $v_0 = (0,0)$, $v_k = (2,3)$. $\quad S = \{\underset{E}{(0,1)}, \underset{N}{(1,0)}\}$



(ENENN) $\quad$ (NNEEN) $\quad$, etc.

TOTAL: go E twice (end at $(2, -)$)

" N thrice (" " $(-, 3)$)

$\rightsquigarrow$ number of paths = arrange $2E$ & $3N$.

$\binom{5}{2,3} = \binom{5}{2} = 10$. (2 categories: E & N)

Permutations of $\{E^2, N^3\}$

Prop: Fix $v = (a_1, \ldots, a_d) \in \mathbb{N}^d$. If $S = \{e_1, \ldots, e_d\}$, then the number of lattice paths from $(0, \ldots 0)$ to v with steps in S is $\binom{a_1 + \cdots + a_d}{a_1, \ldots a_d}$

Proof: Differences $(v_i - v_{i-1})$ correspond to an e_j

Need a_1 differences to be e_1, etc because we must reach $v = \sum_{i=1}^{d} a_i e_i$

a_2 " " e_2

Lattice paths in bijection with $\mathfrak{S}_{\{e_1^{a_1}, \ldots, e_d^{a_d}\}}$, $\sum_{i=1}^{n} a_i$ fixed $\Rightarrow$ Use Thm 3.

Obs: Same will work for any S as long as its linearly independent & $v \in \mathbb{Z}_{\geq 0}$-span of S.

(Exercise)

§ 3. q-binomial coefficients = Gaussian coefficients

$\boxed{\text{Fix } q = \text{parameter}}$

Def: $[n]_q = \dfrac{1-q^n}{1-q} = 1+q+q^2+\cdots+q^{n-1}$ $n \in \mathbb{Z}_{\geq 0}$.

Obs: Setting $q=1$ gives $\underbrace{1+1+1+\cdots+1}_{n \text{ times}} = n$, so we recover n.

Name: $[n]_q$ = q-integer n (q-analog of the integer n)

Def (q-factorials)

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q$$

(replace integers in $n! = n(n-1)\cdots 1$ by q-integers)

with $[0]_q! = 1$.

Obs: Taking $q=1$ recovers $n!$

Def: For $n \in \mathbb{Z}_{\geq 1}$, $a_1, \ldots, a_m \in \mathbb{Z}_{\geq 1}$ with $\sum_{i=1}^{n} a_i = n$ we define the q-multinomial coefficient as $\left[\begin{matrix} n \\ a_1, \ldots, a_m \end{matrix}\right]_q = \dfrac{[n]_q!}{[a_1]_q! \cdots [a_m]_q!}$

(using q-analog of formula from Thm 1).

Obs: Setting $q=1$ recovers $\binom{n}{a_1, \ldots, a_m}$.

Q: Why this construction?

A: • q-analog of the set $\{1\}$ is the finite field $\mathbb{F}_q$ with q-elements ($q = p^k$)

• q- ———————— $[n]$ is the vector space $\mathbb{F}_q^n$.

We'll use q-multinomial coefficients to compute subspaces of $\mathbb{F}_q^n$.