Lecture VI: q-binomial coefficients

$\boxed{q = \text{Fixed parameter}}$

Recall: We have q-analogs of integers, factorials & multinomial coefficients

- $[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+\cdots+q^{n-1} \in \mathbb{Z}[q]$ (q-integer)
- $\begin{cases} [n]_q! = [n]_q[n-1]_q\cdots[1]_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q)}{(1-q)^n} \in \mathbb{Z}[q]. \\ [0]_q! = 1 \end{cases}$ (q-factorial)
- $n, a_1, \ldots, a_m \in \mathbb{Z}_{\geq 1}$, $\sum_{i=1}^m a_i = n \rightsquigarrow \begin{bmatrix} n \\ a_1, \ldots a_m \end{bmatrix}_q = \frac{[n]_q!}{[a_1]_q!\cdots[a_m]_q!}$

(q-multinomial coefficients)

SAGE implementations: q_int(n)
q_factorial(n)
gaussian_multinomial or gaussian_binomial.

Obs: Setting $q=1$ or $\lim_{q\to 1}$ recovers n, $n!$ & $\binom{n}{a_1, \ldots a_m}$, respectively.

⚠ [EC1] uses bold(n) for $[n]_q$ & bold$(n)!$ for $[n]_q!$.

Obs:
- q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$
- $\begin{bmatrix} n \\ a_1, \ldots a_m \end{bmatrix}_q$ is a rational function of q with $\mathbb{Z}$-coefficients.

§1 Properties:

Prop 1: $\begin{bmatrix} n \\ a_1, \ldots a_m \end{bmatrix}_q = \begin{bmatrix} n \\ a_1 \end{bmatrix}_q \begin{bmatrix} n-a_1 \\ a_2 \end{bmatrix}_q \cdots \begin{bmatrix} n-a_1-\cdots-a_{m-1} \\ a_m \end{bmatrix}_q$.

Proof. Express the (RHS) using the definitions & cancel algebraically! □

Prop 2 (Symmetry) $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-n \end{bmatrix}_q$ & in general $\begin{bmatrix} n \\ a_1, \ldots, a_m \end{bmatrix}_q = \begin{bmatrix} n \\ a_{\sigma(1)}, \ldots, a_{\sigma(m)} \end{bmatrix}_q$ for all permutations σ in $\mathfrak{S}_{[m]}$.

Proof: Straight from the definitions.

Examples:
- $\begin{bmatrix} n \\ 1 \end{bmatrix}_q = \begin{bmatrix} n \\ n-1 \end{bmatrix}_q = \frac{[n]_q}{[1]_q} = 1+q+\cdots+q^{n-1}$. ($[1]_q = 1$, $[2]_q = 1+q$ ....)
- $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1$.
- $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4]_q[3]_q[2]_q[1]_q}{[2]_q[1]_q[2]_q[1]_q} = \frac{(1-q^4)}{(1-q)}\frac{(1-q^3)}{(1-q)} \Big/ \frac{(1-q^2)}{(1-q)}\cdot 1 = \frac{(1-q^4)(1-q^3)}{(1-q^2)(1-q)}$
$= \frac{(1-q^4)}{(1-q^2)}\frac{(1-q^3)}{1-q} = (1+q^2)(1+q+q^2) = \boxed{1+q+2q^2+q^3+q^4}$
- $\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_q = 1+q+2q^2+2q^3+2q^4+q^5+q^6$.

Note: We get palindromic polynomials in q with non-negative integer coefficients!

The positivity & polynomiality follow from the q-Pascal recurrence:

Prop 3 $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. for $k \geq 1$, with

initial conditions: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1$, & $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $k<0$ or $k>n$. [same as for usual binomials!]

Note: Usual Pascal recurrence follows by setting $q=1$.

Proof: Straightforward computation:

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = \frac{[n-1]_q!}{[k]_q!\,[n-1-k]_q!} + q^{n-k}\frac{[n-1]_q!}{[k-1]_q!\,[n-k]_q!}$$

$$= \frac{[n-1]_q!}{[k-1]_q!\,[n-1-k]_q!}\left(\frac{1}{[k]_q} + \frac{q^{n-k}}{[n-k]_q}\right)$$

$$= \frac{[n-1]_q!}{[k-1]_q!\,[n-1-k]_q!}\left(\frac{[n-k]_q + q^{n-k}[k]_q}{[k]_q\,[n-k]_q}\right)$$

$$[n-k]_q + q^{n-k}[k]_q = (1+q+\cdots+q^{n-k-1}) + q^{n-k}(1+q+\cdots+q^{k-1})$$

$$= 1+q+\cdots+q^{n-k-1}+q^{n-k}+q^{n-k+1}+\cdots+q^{n-1} = [n]_q$$

Conclude: $\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = \dfrac{[n-1]_q!\,[n]_q}{[k]_q!\,[n-k]_q!} = \begin{bmatrix} n \\ k \end{bmatrix}_q$ ☒

Thm: (1) $\begin{bmatrix} n \\ k \end{bmatrix}_q \in \mathbb{Z}_{\geq 0}[q]$

(2) $\begin{bmatrix} n \\ a_1,\dots,a_m \end{bmatrix}_q \in \mathbb{Z}_{\geq 0}[q]$

} polynomials in q with non-negative coefficients (of degree $k(n-k)$ if $0 \leq k \leq n$)

Proof: Use induction on $(n-k) \in \mathbb{N}_0$ & Prop 3 to prove (1). Then (2) follows from Prop 1.

q-Pascal Triangle:

n \ k	0	1	2	3		
0	1	0	0	. . .		
1	1	1				
2	1	$1+q$	1			
3	1	$1+q+q^2$	$1+q+q^2$	1		
4	1	$1+q+q^2+q^3$	$1+q+2q^2+q^3+q^4$	$1+q+q^2+q^3$	1	. . .

§2 Generalizing k-subsets:

Recall: For each prime power $q=p^m$, there is a finite field with q-elements, denoted by $\mathbb{F}_q$ = Splitting field of $(x^q - x)$ over $\mathbb{F}_p = \mathbb{F}_p[q \text{ roots in } \mathbb{F}_q]$

When q is prime, $\mathbb{F}_q \simeq \mathbb{Z}/p\mathbb{Z}$.

Thm: $\begin{bmatrix} n \\ k \end{bmatrix}_q = \#$ of k-dim'l subspaces of $(\mathbb{F}_q)^n = \bigoplus_{i=1}^{n} \mathbb{F}_q\langle e_i\rangle$.

Here, left hand side is specialized to $q = p^m$ & in (RHS) $q = p^m$

Proof: Uses similar proof techniques from $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

. We count ordered k-Tuples of linearly independent vectors in $(\mathbb{F}_q)^n$ in 2 ways. Call the total number N.

· Write $G_q(n,k) = \{k\text{-subspaces of } (\mathbb{F}_q)^n\}$

$$(1)\ N = \underbrace{(q^n-1)}_{\substack{\text{Pick } 1^{st} \text{ vector } v_1 \\ \text{from } \mathbb{F}_q^n \setminus \{0\}}} \underbrace{(q^n-q)}_{\substack{\text{Pick } 2^{nd} \text{ vector } v_2 \\ \text{from } \mathbb{F}_q^n \setminus \mathbb{F}_q\langle v_1\rangle}} \underbrace{(q^n-q^2)}_{\substack{\text{Pick } 3^{rd} \text{ vector } v_3 \\ \text{from } \mathbb{F}_q^n \setminus (\mathbb{F}_q\langle v_1\rangle \oplus \mathbb{F}_q\langle v_2\rangle)}} \cdots\cdots \underbrace{(q^n-q^{k-1})}_{\substack{\text{Pick } k^{th} \text{ vector } v_k \\ \text{from } \mathbb{F}_q^n \setminus \bigoplus_{i=1}^{k-1} \mathbb{F}_q\langle v_i\rangle \\ \text{size} = q^{k-1}}}$$

$$(2)\ N = |G_q(n,k)| \cdot (q^k-1)(q^k-q)\cdots\cdots(q^k-q^{k-1})$$

From each subspace V in $G_q(n,k)$, pick a basis $\{v_1, \dots v_k\}$

$$\underset{\mathbb{F}_q^k}{\overset{}{\cong}} \qquad \text{basis count} = \begin{cases} \cdot\ q^k-1 \text{ choices for } v_1 \\ \cdot\ q^k-q \quad \text{"} \quad \text{"}\ v_2 \quad (v_2 \in \mathbb{F}_q^k \setminus \mathbb{F}_q\langle v_1\rangle) \\ \quad\vdots \\ \cdot\ q^k-q^{k-1} \quad \text{"} \quad \text{"}\ v_k \quad (v_k \in \mathbb{F}_q^k \setminus \bigoplus_{i=1}^{k-1} \mathbb{F}_q\langle v_i\rangle) \end{cases}$$

Since basis determine subspaces, we equate both expressions to get

$$|G_q(n,k)| = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})} = \frac{(1-q^n)(q-q^n)\cdots(q^{k-1}-q^n)}{(1-q^k)(q-q^k)\cdots(q^{k-1}-q^k)}$$

$$= \frac{(1-q^n)\,\cancel{q}(1-q^{n-1})\,\cancel{q^2}(1-q^{n-2})\cdots \cancel{q^{k-1}}(1-q^{n-k+1})}{(1-q^k)\,\cancel{q}(1-q^{k-1})\,\cancel{q^2}(1-q^{k-2})\cdots \cancel{q^{k-1}}(1-q)}$$

$$= \frac{\frac{(1-q^n)}{(1-q)}\frac{(1-q^{n-1})}{(1-q)}\frac{(1-q^{n-2})}{(1-q)}\cdots\frac{(1-q^{n-k+1})}{(1-q)}}{\frac{(1-q^k)}{(1-q)}\frac{(1-q^{k-1})}{(1-q)}\frac{(1-q^{k-2})}{(1-q)}\cdots\frac{(1-q)}{(1-q)}} = \frac{[n]_q[n-1]_q\cdots[n-k+1]_q}{[k]_q[k-1]_q\cdots[1]_q}$$

$$= \begin{bmatrix} n \\ k \end{bmatrix}_q . \quad \blacksquare$$