Lecture VII: Inversions & q-factorial numbers

GOAL: Look at $[n]_q!$ more closely & characterize the coefficients of these polynomials in $\mathbb{Z}[q]$.

§1 Inversions (See also Problem 1 in HW2)

Definition: Given a permutation $\sigma \in \mathfrak{S}_n = S_n$, an inversion of σ is a pair $(\sigma(i), \sigma(j))$ where $i<j$ & $\sigma(i) > \sigma(j)$.

We let $\operatorname{inv}(\sigma)$ be the number of inversions of the permutation $\sigma : [n] \xrightarrow{1\text{-to-}1} [n]$.

Example: $\mathfrak{S}_3$: $\sigma = (\sigma(1), \dots, \sigma(n))$

$\operatorname{inv}(123) = 0$ $\quad\operatorname{inv}(213) = 1$ $\quad\operatorname{inv}(312) = 2$

$\operatorname{inv}(132) = 1$ $\quad\operatorname{inv}(231) = 2$ $\quad\operatorname{inv}(321) = 3$

• Write the o.g.f. for inv over $\mathfrak{S}_n$, using q as the variable:

$$F_n(q) := \sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{inv}(\sigma)} = \sum_{i=0}^{\infty} a_i\, q^i \quad \text{where } a_i = \#\{\sigma \in \mathfrak{S}_n \mid \operatorname{inv}(\sigma) = i\} \text{ for all } i = 0, 1, \dots$$

Example: $\mathfrak{S}_3$:

$$q^0 + 2q^1 + 2q^2 + q^3 = (1+q)(1+q+q^2) = [2]_q\,[3]_q = [3]_q!$$

σ : (123); (132), (213); (231), (312); (321) $\qquad$ $[1]_q = 1$

This is a general result!

Thm (MacMahon): Fix n positive integer. Then:

$$\boxed{\sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{inv}(\sigma)} = [n]_q!}$$

We aim for a bijective proof, writing $[n]_q! = [n]_q [n-1]_q \cdots [1]_q$ & interpreting each $q^{\operatorname{inv}(\sigma)} = q^{a_1} \cdot q^{a_2} \cdots q^{a_n}$ for a unique choice $(a_1, \dots, a_n)$ where a_i corresponds to a monomial q^{a_i} in $[n+1-i]_q$. So, the monomial $q^{a_1+\dots+a_n}$ in (RHS) has coefficient 1.

More explicit, we need a bijection:

$$I : \mathfrak{S}_n \longrightarrow T_n \quad \text{where } T_n = \{(a_1, \dots, a_n) : 0 \le a_i \le n-i\} = \underbrace{[0,n-1]}_{\text{monomials in } [n]_q} \times \underbrace{[0,n-2]}_{\text{monomials in } [n-1]_q} \times \cdots \times \underbrace{[0,0]}_{\text{monomials in } [1]_q}$$

satisfying $\operatorname{inv}(\sigma) = a_1 + \dots + a_n$ when $I(\sigma) = (a_1, \dots, a_n)$.

Def: $I(\sigma)$ is called the inversion table for σ.

Proof of Theorem: Assume the bijection I exists. Then:

$$\sum_{\sigma \in \mathfrak{S}_n} q^{inv(\sigma)} = \sum_{(a_1,\ldots,a_n)\in T_n} q^{a_1+\cdots+a_n} = \sum_{a_1=0}^{n}\sum_{a_2=0}^{n-1}\cdots\sum_{a_n=0}^{0} q^{a_1+a_2+\cdots+a_n} \underset{\text{Distributive Law}}{=}$$

$$=\left(\sum_{a_1=0}^{n-1} q^{a_1}\right)\left(\sum_{a_2=0}^{n-2} q^{a_2}\right)\cdots\cdots\left(\sum_{a_{n-1}=0}^{1} q^{a_{n-1}}\right)\left(\sum_{a_n=0}^{0} q^{a_n}\right)$$

$$=(1+q+\cdots+q^{n-1})\,(1+q+\cdots+q^{n-2})\;\cdots\cdots\;(1+q)\cdot(1) \;=[n]_q[n-1]_q\cdots[1]_q=[n]_q! \quad \boxed{}$$

§2: Building Inversion Tables: $|T_n| = |\mathfrak{S}_n| = n!$

Rather than building I, we construct its inverse bijection $I^{-1}: T_n \longrightarrow \mathfrak{S}_n$ as an algorithm: We place each $\{n,\ldots,n-i+1\}$ in a word $w^{(i)}$ of length i, relative locations dictated by elements in T_n.

- Input: $\underline{a}=(a_1,\ldots,a_n)\in T_n = [0,n-1]\times[0,n-2]\times\cdots\times[0,0]$
- Output: A permutation $\sigma = I^{-1}(\underline{a})$ in word format of length n.

- Start: $i=1$ $\quad w^{(1)} = n$ $\quad$ ($a_n = 0$ .. ALWAYS)
- For $i=2,\ldots,n$: Create $w^{(i)}$ from $w^{(i-1)}$ by inserting $(n-i+1)$ into $w^{(i-1)}$ so that there are a_{n-i} elements in $w^{(i)}$ to the left of $(n-i+1)$.
- Return: $w^{(n)}$

Example: $\underline{a} = (0,3,3,3,2,1,0) \in T_7$ $\quad$ ($a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7$) $\quad n=7$ $\longleftarrow$

$w^{(1)} = 7$

$w^{(2)} = 7\underline{6}$ $\quad$ (1 element to the left of 6) $\quad$ [if $a_6=0$, then $w^{(2)}=67$]

$w^{(3)} = 76\underline{5}$ $\quad$ (2 ——— of 5) $\quad$ [if $a_5=1$, then $w^{(3)}=756$]

$w^{(4)} = 765\underline{4}$ $\quad$ (3 ——— 4)

$w^{(5)} = 765\underline{3}4$ $\quad$ (3 ——— 3)

$w^{(6)} = 765\underline{2}34$ $\quad$ (3 ——— 2)

$I^{-1}(\underline{a}) = w^{(7)} = \underline{1}765234$ $\quad$ (0 ——— 1)

Proposition 1: $I^{-1}: T_n \longrightarrow \mathfrak{S}_n$ is a bijection

Proof: $|T_n| = |\mathfrak{S}_n| = n!$, so it's enough to prove the map is injective.

If $\underline{a}, \underline{b} \in T_n$ are different, then $a_i \neq b_i$ for some i. Pick i maximal with that property. Then $w_{\underline{a}}^{(i)} \neq w_{\underline{b}}^{(i)}$ since $(n-i+1)$ is placed in these 2 words in a different order relative to $n, n-1, \ldots, n-i+1$. [Note: $i<n$ because $a_n=b_n=0$]

From this: $I^{-1}(\underline{a}) \neq I^{-1}(\underline{b})$. $\boxed{}$

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Proposition 2: $\mathrm{Inv}(I^{-1}(\underline{a})) = a_1 + \cdots + a_n$

Proof $\underbrace{\mathrm{Inv}(I^{-1}(\underline{a}))}_{=\sigma}$ = # inversions of σ having 1 in 2nd position.

+ # ______ 2 ______.

+ ⋮

______ n ______.

• j in 2nd position of an inversion means $(\underset{\sigma(p)}{\overset{\shortparallel}{k}}, \underset{\sigma(p')}{\overset{\shortparallel}{j}})$ with $p < p'$ & $k > j$.

So # inversions of σ with j in 2nd position is exactly the number of elements k to the left of j in σ with $k > j$.

This is the number of elements to the left of j in the partial word $w^{(n-j)}$.

[Write $j = n - i$ so $i = n - j$] This is $a_{n-(n-j)} = a_j$

So $\mathrm{Inv}(\sigma) = a_1 + a_2 + \cdots + a_n$ as we wanted. $\square$

Example above: $\underline{a} = (0,3,3,3,2,1,0) \in T_7$ & $\sigma = I^{-1}(\underline{a}) = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 7 & 6 & 5 & 2 & 3 & 4 \end{array} \in \mathfrak{S}_7$

$a_1 = 0$ elements to the left of $1 = \sigma(1)$ that are larger than 1.

$a_2 = 3$ ______ $2 = \sigma(5)$ ______ 2 : 7, 6 & 5.

$a_3 = 3$ ______ $3 = \sigma(6)$ ______ 3 : 7, 6 & 5.

$a_4 = 3$ ______ $4 = \sigma(7)$ ______ 4 : 7, 6 & 5.

$a_5 = 2$ ______ $5 = \sigma(4)$ ______ 5 : 7 & 6.

$a_6 = 1$ ______ $6 = \sigma(3)$ ______ 6 : 7.

$a_7 = 0$ ______ $7 = \sigma(2)$ ______ 7.

Another example: $\sigma = 21354 \in \mathfrak{S}_5 \xrightarrow{I} \underline{a} = (1,0,0,1,0)$

$$I^{-1}(\underline{a}) =: \underset{(a_5=0)}{w^{(1)} = 5} \rightsquigarrow \underset{(a_4=1)}{w^{(2)} = 5\underline{4}} \rightsquigarrow \underset{(a_3=0)}{w^{(3)} = \underline{3}54} \rightsquigarrow \underset{(a_2=0)}{w^{(4)} = \underline{2}354} \rightsquigarrow \underset{(a_1=1)}{w^{(5)} = 2\underline{1}354}$$

as expected!