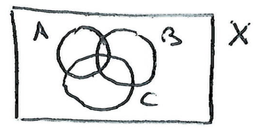
Lecture VIII: Intro to Sieve-Methods & Inclusion-Exclusion

2 Sieve Methods: (1) Principle of Inclusion-Exclusion (adjust overcounting)

(2) Sieve unwanted elements by suitable weighting (Principle of Involution & Gessel-Viennot's Lemma)

§1. Inclusion-Exclusion for finite sets:



Example: $A, B, C \subseteq X$ sets, $|X| < \infty$.

. **Sum Rule**: $|A \cup B \cup C| = |A| + |B| + |C|$ whenever they are pairwise disjoint.

. **Q**: What to do if we have common elements? $|A|+|B|+|C|$ overcounts!

A: $|A \cup B \cup C| = \underbrace{|A| + |B| + |C|}_{\substack{\text{overcounts } |A\cap B| \text{ once} \\ |B\cap C| \text{ "} \\ |A \cap C| \text{ "}}} - |A\cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

b/c we've cancelled $|A\cap B\cap C|$'s contribution

Equivalently: $|X \setminus (A \cup B \cup C)| = |X| - |A| - |B| - |C| + |A\cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$.

Prop 1: $A_1, A_2, \ldots, A_n \subset X$ subsets $|X| < \infty$. Then:

$$|X \setminus \bigcup_{i=1}^{n} A_i| = |X| - \sum_{i=1}^{n} |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|$$

Proof: Pick $x \in X$ & see how many times it's counted on both sides.

(1) If $x \notin \bigcup_{i=1}^{n} A_i$, then x only appears in $|X|$ on (RHS) & also once in (LHS).

(2) If $x \in \bigcup_{i=1}^{n} A_i$, Write $x \in \bigcap_{j=1}^{m} A_{i_j}$ with m maximal (ie $x \notin A_\ell$ for $\ell \neq i_j$) $(m \geq 1)$

- x is not counted on (LHS).
- On (RHS), x is counted only when ONLY $A_{i_1}, \ldots A_{i_m}$ are involved $\leadsto \binom{m}{k}$ for each intersection of k subsets

So total count is $= 1 - \binom{m}{1} + \binom{m}{2} - \binom{m}{3} + \cdots + (-1)^m \binom{m}{m}$

$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} \underset{\substack{\uparrow \\ \text{Binomial} \\ \text{Theorem}}}{=} (1+x)^m \Big|_{x=-1} = 0 \quad \text{because } m \geq 1.$$

∎

Next, we'll provide many generalizations to the counting method by "Inclusion-Exclusion".

§2: Inclusion-Exclusion for Properties:

- Set X = universe $|X|<\infty$
- $E=\{e_1,\dots,e_n\}$ set of properties that each $x\in X$ may or may not satisfy.
- $A_i=\{x\in X \mid x$ satisfies $e_i\}$ $i=1,\dots,n$

Then:
- $X\setminus\bigcup_{i=1}^n A_i$ = elements satisfying none of $e_1,\dots,e_n$.
- $A_{i_1}\cap\dots\cap A_{i_s}=\{x\in X \mid x$ satisfies $e_{i_1},\dots,e_{i_s}\}$

Def: $N_{\supseteq T}:=\#\{x\in X \mid x$ satisfies properties in $T\}$ for $T\subseteq E$

$N_{=T}:=\#\{x\in X \mid x$ —— precisely properties in $T\}$

Principle of Inclusion-Exclusion (P.IE) X, E as above. Then

$$N_{=\emptyset}=\sum_{T\subseteq E}(-1)^{|T|}N_{\supseteq T}=\sum_{k=0}^n(-1)^k\sum_{T:|T|=k}N_{\supseteq T}.$$

Special case: E homogeneous, meaning $N_{\supseteq T}$ only depends on $|T|$, call it $N_{\geq|T|}$.

If so: $N_{=\emptyset}=\sum_{k=0}^n(-1)^k N_{\geq|T|}\binom{n}{k}$

Obs: If $N_{\supseteq T}$ only depends on $|T|$, so does $N_{=T}$. This is a consequence of:

Generalization 1 of PIE: X, E as above & $B\subseteq E$. Then:

(1) $N_{=B}=\sum_{E\supseteq T\supseteq B}(-1)^{|T|-|B|}N_{\supseteq T}$. & (2) $N_{\supseteq B}=\sum_{T\supseteq B}N_{=T}$

Proof: By def of $N_{\supseteq}$ & $N_=$: $N_{\supseteq B}=\sum_{T\supseteq B}N_{=T}$. The claim (1) will follow from (2) if we can "invert" this linear map with variables $N_{=T}$. [This will be done more generally later]

Write $X_{\supseteq T}=\bigcap_{i\in T}A_i$ & $X_{=T}=\{x\in X \mid x$ satisfies exactly $T\}$

To show: $N_{=B}\underset{\text{"Caseman"}}{=}\sum_{a\in X_{=B}}1\ \boxed{=}\sum_{B\subseteq T\subseteq E}(-1)^{|T\setminus B|}\sum_{a\in X_{\supseteq T}}1\underset{\text{"Caseman"}}{=}\sum_{B\subseteq T\subseteq E}(-1)^{|T|-|B|}N_{\supseteq T}$

Next, we prove the middle equality, by checking contribution of each $x\in X$ on each side. We have 3 cases:

CASE 1: $x \in X \setminus X_{\geq B}$, then x contributes nothing to either side

CASE 2: $x \in X_{=B}$, then x contributes exactly ones on each side
(On (RHS): only contribution comes from $T=B$)

CASE 3: $x \in X_{\geq B} \setminus X_{=B}$. Then x is not in (LHS). & $\exists$ unique $\tilde{T}$ with

$B \subsetneq \tilde{T} \subseteq E$ & $x \in X_{=\tilde{T}}$. For any $\tilde{T} \geq T$, x appears in $\sum_{x \in X_{\geq T}}$ & it doesn't appear for $T \supsetneq \tilde{T}$.

So total count for x in (RHS) $:= \displaystyle\sum_{\tilde{T} \geq T \geq B} (-1)^{|T-B|} \underset{k=|T-B|}{=} \sum_{k=0}^{|\tilde{T}-B|} (-1)^k \binom{|\tilde{T} \setminus B|}{k} \underset{\text{Binomial Theorem}}{=} (1+x)^{|\tilde{T} \setminus B|}\Big|_{x=-1} = 0$ b/c $\tilde{T} \supsetneq B$ ∎

§3 Examples:

Ex 1 (Problem 3 HW1) $\varphi(n) := \#\{k \mid 1 \leq k \leq n \ \gcd(k,n)=1\}$ (Euler's φ-function)

Want to recover $\varphi(n)$ from $\sum_{d \mid n} \varphi(d) = n$. using P.I.E

Write $n = p_1^{a_1} \cdots p_t^{a_t}$ prime decomposition.

$X = \{1, 2, \ldots n\}$

$e_i =$ "$p_i \mid x$" property for $x \in X$.

Clear $N_{=\emptyset} = \varphi(n)$. & $N_{\geq T} = \#\{x \in X : \prod_{e_i \in T} p_i \mid x\} = \dfrac{n}{\prod_{e_i \in T} p_i}$.

PIE: $N_{=\emptyset} = \displaystyle\sum_{k=0}^{n} (-1)^k \sum_{T : |T|=k} N_{\geq T}$

$$\varphi(n) = \underset{(k=0)}{n} - \underset{(k=1)}{\sum_{i=1}^{t} \frac{n}{p_i}} + \underset{(k=2)}{\sum_{i<j} \frac{n}{p_i p_j}} - \cdots + \underset{(k=t)}{(-1)^t \frac{n}{p_1 \cdots p_t}}$$

$$= n \left(1 - \sum_{i=1}^{t} \frac{1}{p_i} + \sum_{i<j} \frac{1}{p_i p_j} - \cdots + (-1)^t \frac{1}{p_1 \cdots p_t} \right)$$

$$= n \prod_{i=1}^{t} \left(1 - \frac{1}{p_i} \right)$$

Example: $n = 20 = 2^2 \cdot 5$ gives $\varphi(20) = 20 \cdot \frac{1}{2} \cdot \frac{4}{5} = 8$. $(\#\{1, 3, 7, 9, 11, 13, 17, 19\})$

Consequence: $\varphi(mn) = \varphi(m)\varphi(n)$ when m & n are relatively prime.

Ex 2: (Lecture III) # of non-neg integer solutions to $\sum_{i=1}^{k} x_i = n$ is $\binom{n+k-1}{k-1}$ (weak k-compositions of n).

Q: How many satisfy $x_i < s$ for all i? (Here: s is fixed)

A: Use PIE

$X = \{$weak k-compositions of n$\}$

$e_i =$ "$x_i \geq s$" for $x = (x_1, x_2, \ldots, x_k) \in X$.

$E = \{e_1, \ldots, e_k\}$ homogeneous $[X_{\geq T} := \{x \mid x_i \geq s \ \forall e_i \in T\}$ & $N_{\geq T}$ only depends on $|T|$ (rearrange $\underline{x}$ so that first $|T|$ terms are $\geq s$) $]$

- $N_{=\emptyset}$ = what we wanted to count
- $N_{\geq T} \underset{|T|=j}{=} N_{\geq\{1,\ldots,j\}} = \#\left\{ \underline{x} : \begin{array}{l} x_1 + \cdots + x_k = n \\ x_1, \ldots, x_j \geq s \end{array} \right\}$

Set $y_i = \begin{cases} x_i - s & \text{for } i = 1, \ldots, j \\ x_i & \text{for } i = j+1, \ldots, n \end{cases}$ Then $\left\{ \begin{array}{l} x_1 + \cdots + x_k = n \\ x_1, \ldots x_j \geq s \\ x_1 \cdots x_k \geq 0 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} y_1 + \cdots + y_k = n - js \\ y_1 \cdots y_k \geq 0 \end{array} \right\}$ weak k-comp of $n-js$.

So $N_{\geq T} = \binom{n-js+k-1}{k-1}$ if $|T| = j$

PIE gives (homogeneous): $N_{=\emptyset} = \sum_{j=0}^{k} (-1)^j \binom{n-js+k-1}{k-1}\binom{k}{j}$

Special case: $s=1 \rightsquigarrow$ want $x_i = 0$ for all i & $x_1 + \cdots + x_k = n$. This gives the identity:

$$\sum_{j=0}^{k} (-1)^j \binom{n+k-j-1}{k-1}\binom{k}{j} = N_{=\emptyset} = \begin{cases} 0 & \text{if } n > 0 \\ 1 & \text{" } n = 0 \end{cases}$$

General: $\sum_{j=0}^{m} (-1)^j \binom{m}{j} c_j = ??$ The value can be computed by PIE if we interprete c_j as $N_{\geq j}$ for some homogeneous set of properties. **A**: $N_{=\emptyset}$.

Ex 3 Determine $\sum_{j=0}^{m} (-1)^j \binom{m}{j} \underbrace{\binom{n-j}{r}}_{=:c_j}$ for $m \leq r \leq n$.

Soln: $c_0 = \binom{n}{r} = |X| \rightsquigarrow X = \binom{[n]}{r}$

We fix $M = \{1, \ldots, m\} = [m]$ & define the properties $E = \{e_1, \ldots, e_m\}$ with $e_i :=$ "$i \notin A$", for any element $A \in X$. for $i = 1, \ldots m$

Then: $N_{\geq T} = \{ A \in \binom{[n]}{r} \;\; i \notin A \text{ for all } e_i \in T \}$

$\qquad = \{ A \in \binom{[n]-|T|}{r} \} = \binom{n-|T|}{r}$ E is homogeneous set of properties

$N_{=\emptyset} = \{ A \in \binom{[n]}{r} : [m] \subseteq A \} = \binom{n-m}{r-m}$

So $\displaystyle\sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{n-j}{r} \underset{\text{PIE}}{=} N_{=\emptyset} = \binom{n-m}{r-m}$ □